Nonstandard Analysis on $\widetilde{\mathbb{R}}$

Transfer, standard part, compactness, and Hilbert spaces

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Outline

This lecture: bridging two worlds

Goal. Develop an analysis framework on $\widetilde{\mathbb{R}}$ that recovers classical real analysis via systematic translation.

Two worlds.

- $\mathbb{R}$: the standard reals — Archimedean, complete.
- $\widetilde{\mathbb{R}}$: the extended reals of the grain program — containing infinitesimals and infinite elements.

Four steps.

1. **Transfer principle.** Logical bridge between $\mathbb{R}$ and $\widetilde{\mathbb{R}}$.
2. **Standard part.** The operator st extracts classical content from $\widetilde{\mathbb{R}}$.
3. **Compactness.** Nonstandard characterization and its classical consequences.
4. **Hilbert spaces.** Extension to metric spaces, Banach spaces, and Hilbert spaces with key theorems.

Spirit. Nonstandard analysis simplifies proofs by replacing ε-δ arguments with intuitive notions of « infinitely close ». The grain program gives this approach a constructive geometric foundation.

Setup: formal language of ordered fields

Definition (Language $\mathcal{L}$)

The first-order language of ordered fields:

- Constants: 0, 1.
- Binary operations: $+$, $\cdot$.
- Binary relation: $<$.
- Logical connectives: $\neg$, $\wedge$, $\vee$, $\Rightarrow$.
- Quantifiers: $\forall$, $\exists$ (always bounded).

Formulas. Built recursively from atomic formulas (polynomial equations and inequalities) using logical operations.

Crucial restriction. Quantifiers must range over a *specified set* (here, $\mathbb{R}$ or $\widetilde{\mathbb{R}}$).

Transferable formulas

Definition (Transferable formula)

A formula φ is *transferable* if it can be expressed in $\mathcal{L}$ *without unbounded quantifiers*, using only:

- Atomic formulas with polynomial expressions.
- Bounded quantifiers $\forall x \in S$, $\exists x \in S$ where S is a definable set.

Dual interpretations. A transferable formula φ has two interpretations:

- $\varphi^{\mathbb{R}}$: evaluated in $\mathbb{R}$.
- $\varphi^{\widetilde{\mathbb{R}}}$: evaluated in $\widetilde{\mathbb{R}}$.

The transfer theorem

Theorem (Transfer)

For any transferable formula φ:

$$\boxed{\varphi^{\mathbb{R}} \iff \varphi^{\widetilde{\mathbb{R}}}.}$$

That is: the formula holds in $\mathbb{R}$ if and only if it holds in $\widetilde{\mathbb{R}}$.

Reading. The transfer principle is a *logical bridge*: it lets us translate properties freely between the two worlds, provided the formulas are properly formulated.

Powerful consequence. Classical theorems can be proved using nonstandard tools (infinitesimals, infinite elements), and the result transfers back to $\mathbb{R}$.

Examples and counter-examples

Transferable formulas.

- Commutativity: $\forall x, y\ (x + y = y + x)$.
- Field axioms: existence of inverse, distributivity, etc.
- Quadratic formula: solutions of $ax^2 + bx + c = 0$.

Non-transferable formulas.

- Archimedean property: « $\forall x \in \mathbb{R}\ \exists n \in \mathbb{N}$ such that $n > x$ ». Fails in $\widetilde{\mathbb{R}}$: no finite n exceeds n_∞.
- Completeness: « every bounded sequence has a least upper bound ». Involves second-order quantification.

Diagnosis. Non-transferable properties involve the *external structure* of $\mathbb{R}$ (Archimedean order, completeness), not its first-order algebraic content.

Nonstandard characterization of limits

Proposition (Nonstandard limit)

Let (u_n) be a sequence of standard reals. Then

$$\lim_{n \to \infty} u_n = L \iff \forall n \in \widetilde{\mathbb{N}} \setminus \mathbb{N},\ u_n \approx L,$$

where $\approx$ denotes infinitesimal proximity.

Reading. A sequence converges to L iff all its infinite terms are infinitely close to L.

Advantage. This replaces the classical ε-N_ε definition with a direct condition on infinite indices. Proofs become significantly more intuitive.

Halo, monad, limited elements

Definitions

For $x, y \in \widetilde{\mathbb{R}}$:

- $x \approx y$ (infinitesimally close) if $|x - y|$ is infinitesimal.
- *Halo* of x: $\mathrm{hal}(x) = \{y \in \widetilde{\mathbb{R}} : y \approx x\}$.
- *Monad* of a standard x: $\mathrm{hal}(x) \cap \widetilde{\mathbb{R}}$.
- x is *limited* if $\exists r \in \mathbb{R}^+$ such that $|x| < r$.

Equivalence. The relation $\approx$ is an equivalence relation on $\widetilde{\mathbb{R}}$: reflexive, symmetric, transitive.

Geometric picture. Each standard real has a « cloud » of infinitesimally close points around it (its monad). Limited elements have a finite reference point.

The standard part theorem

Theorem (Existence and uniqueness of st)

For every limited $x \in \widetilde{\mathbb{R}}$, there exists a unique $\mathrm{st}(x) \in \mathbb{R}$ such that:

$$x \approx \mathrm{st}(x).$$

The element $\mathrm{st}(x)$ is the *standard part* of x.

The standard part operator.

$$\mathrm{st} : \widetilde{\mathbb{R}}_{\mathsf{lim}} \to \mathbb{R},$$

where $\widetilde{\mathbb{R}}_{\mathsf{lim}}$ is the ring of limited elements.

Geometric reading. st projects a limited hyper-real onto its closest standard real, by collapsing its halo to a point.

Algebraic compatibility

Theorem (st is a ring morphism)

For limited $x, y \in \widetilde{\mathbb{R}}_{\text{lim}}$:

- $\mathrm{st}(x + y) = \mathrm{st}(x) + \mathrm{st}(y)$.
- $\mathrm{st}(x \cdot y) = \mathrm{st}(x) \cdot \mathrm{st}(y)$.
- $\mathrm{st}(0) = 0$, $\mathrm{st}(1) = 1$.
- If $\mathrm{st}(y) \neq 0$: $\mathrm{st}(x/y) = \mathrm{st}(x)/\mathrm{st}(y)$.

The kernel of st is the ideal of infinitesimals.

Consequence. The quotient $\widetilde{\mathbb{R}}_{\text{lim}}/(\text{infinitesimals})$ is canonically isomorphic to $\mathbb{R}$.

Order compatibility (weak). $x \leq y \Rightarrow \mathrm{st}(x) \leq \mathrm{st}(y)$. The converse fails for strict inequality (infinitesimal differences vanish).

Application: derivative and integral

Nonstandard derivative.

$$f'(a) := \mathrm{st}\left(\frac{f(a+\varepsilon) - f(a)}{\varepsilon}\right)$$

for any infinitesimal $\varepsilon \neq 0$, when the standard part exists and is independent of ε.

Theorem (Equivalence with classical derivative)

f is classically differentiable at a iff the nonstandard derivative exists, and they coincide.

Nonstandard integral. For f continuous on $[a, b]$:

$$\int_a^b f(x)\,dx := \mathrm{st}\left(\sum_{k=0}^{N-1} f(x_k) \cdot \Delta x\right),$$

where N is infinite, $\Delta x = (b-a)/N$ is infinitesimal, and $x_k = a + k\Delta x$.

Equivalence with Riemann integral. Both define the same value for f continuous.

Nonstandard continuity

Definition (Nonstandard continuity)

$f : \mathbb{R} \to \mathbb{R}$ is *nonstandard continuous* at $a \in \mathbb{R}$ if

$$\forall x \in \widetilde{\mathbb{R}} : \quad x \approx a \ \Rightarrow \ f(x) \approx f(a).$$

Theorem (Equivalence)

f is nonstandard continuous at a iff it is classically continuous at a.

Pedagogical advantage. The nonstandard characterization expresses continuity as « infinitely close inputs give infinitely close outputs » — a direct, intuitive condition without ε-δ.

Operational power. Combined with the transfer principle, this yields short proofs of classical continuity results.

Robinson's characterization

Theorem (Robinson)

A subset $K \subset \mathbb{R}$ is *compact* iff every element of its nonstandard extension $\widetilde{K}$ is *nearly standard*:

$$K \text{ compact} \iff \forall x \in \widetilde{K},\ \exists s \in K : x \approx s.$$

Reading. Compactness = « no escape to infinity, no escape into uncovered region »: every nonstandard point in $\widetilde{K}$ has a standard companion in K.

Conceptual advantage. Replaces the classical definitions (closed and bounded, finite subcovers, sequential compactness) with a single direct condition.

Strategy. Apply this characterization to prove four classical theorems with elegant nonstandard arguments.

Consequence 1: Bolzano-Weierstrass

Theorem (Bolzano-Weierstrass)

Every bounded sequence (u_n) in $\mathbb{R}$ has a convergent subsequence.

Nonstandard proof.

- By boundedness, all u_n lie in a compact interval K.
- Pick any infinite index $\nu \in \widetilde{\mathbb{N}} \setminus \mathbb{N}$: $u_\nu \in \widetilde{K}$.
- By Robinson's theorem, $u_\nu \approx L$ for some $L \in K$.
- This L is a limit of a subsequence (by infinite indices clustering at L).

Compare. Classical proof requires the nested interval lemma and a recursive subsequence construction — significantly more involved.

Consequence 2: Heine (uniform continuity)

Theorem (Heine)

A continuous function on a compact set is *uniformly continuous*.

Nonstandard characterization of uniform continuity.

$$\forall x, y \in \widetilde{K} : \quad x \approx y \;\Rightarrow\; f(x) \approx f(y).$$

Proof.

- Let $x, y \in \widetilde{K}$ with $x \approx y$.
- By Robinson: $x \approx s$ for some $s \in K$, and $y \approx x \approx s$.
- By continuity at s: $f(x) \approx f(s)$ and $f(y) \approx f(s)$.
- Therefore $f(x) \approx f(y)$. $\square$

Reading. The compactness of K guarantees that nonstandard points have standard companions, transferring local continuity to a uniform property.

Consequences 3 & 4: extreme values and IVT

Theorem (Extreme values)

A continuous function on a compact set attains its supremum and infimum.

Sketch. Take a sequence approaching the supremum, pick an infinite index, apply Robinson — the resulting standard point achieves the maximum.

Theorem (Intermediate value theorem)

A continuous function on $[a, b]$ takes all values between $f(a)$ and $f(b)$.

Sketch. Bisect the interval, follow the value, take standard part of the limit. Compactness ensures no « missing » values.

General pattern. Compactness theorems become nearly trivial once Robinson's characterization is in hand.

Metric spaces: nonstandard extension

Definition (Nonstandard extension of metric space)

Let (E, d) be a metric space. Its *nonstandard extension* $\widetilde{E}$ inherits a metric $\widetilde{d}$ with values in $\widetilde{\mathbb{R}}$.
For $x, y \in \widetilde{E}$:

$$x \approx y \quad \Longleftrightarrow \quad \widetilde{d}(x, y) \text{ is infinitesimal.}$$

Nonstandard convergence.

$$u_n \to L \quad \Longleftrightarrow \quad \forall n \in \widetilde{\mathbb{N}} \setminus \mathbb{N},\ u_n \approx L.$$

Nonstandard continuity.

$$f \text{ continuous at } a \quad \Longleftrightarrow \quad \forall x \in \widetilde{E},\ x \approx a \Rightarrow f(x) \approx f(a).$$

Cauchy sequences and completeness

Nonstandard characterization of Cauchy

(u_n) is Cauchy iff

$$\forall n, m \in \widetilde{\mathbb{N}} \setminus \mathbb{N} : \quad u_n \approx u_m.$$

Reading. A sequence is Cauchy iff its infinite terms are all pairwise infinitely close.

Theorem (Nonstandard characterization of completeness)

(E, d) is complete iff every limited element of $\widetilde{E}$ has a standard part in E.

Parallel with $\widetilde{\mathbb{R}}$. The completeness of $\mathbb{R}$ is captured by the existence of st on limited elements. The same pattern generalizes to all complete metric spaces.

Banach spaces and standard part

Definitions

- *Normed vector space* $(E, \|\cdot\|)$: vector space with a norm.
- *Banach space*: complete normed vector space.

Proposition (Standard part in Banach)

Let E be a Banach space. For every limited $x \in \widetilde{E}$, there exists a unique $\mathrm{st}(x) \in E$ such that $x \approx \mathrm{st}(x)$.

Properties of st**.**

- st is linear: $\mathrm{st}(\alpha x + \beta y) = \alpha \mathrm{st}(x) + \beta \mathrm{st}(y)$ for α, β standard scalars.
- Compatible with norm: $\mathrm{st}(\|x\|) = \|\mathrm{st}(x)\|$.

Lifting st **from** $\mathbb{R}$**.** The standard part operator generalizes naturally from $\widetilde{\mathbb{R}}$ to any complete normed space.

Pre-Hilbert spaces

Definition (Pre-Hilbert space)

A *pre-Hilbert space* is a vector space E equipped with an inner product $\langle \cdot, \cdot \rangle : E \times E \to \mathbb{R}$ satisfying:

- Linearity in the first argument.
- Symmetry: $\langle x, y \rangle = \langle y, x \rangle$.
- Positive definiteness: $\langle x, x \rangle \geq 0$, with equality iff $x = 0$.

The norm induced is $\|x\| := \sqrt{\langle x, x \rangle}$.

Theorem (Cauchy-Schwarz)

For all $x, y \in E$:

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|.$$

Nonstandard reading. The inner product extends to $\widetilde{E} \times \widetilde{E} \to \widetilde{\mathbb{R}}$, preserving the inequality and its standard part.

Definition (Hilbert space)

A *Hilbert space* is a complete pre-Hilbert space.

Equivalent characterizations.

- Pre-Hilbert + complete (definition).
- Banach space with inner product compatible with the norm.
- Every Cauchy sequence converges, and the parallelogram law holds: $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

Examples.

- Euclidean space $\mathbb{R}^n$ with standard inner product.
- $\ell^2 = \{(x_n) : \sum |x_n|^2 < \infty\}$.
- $L^2(\Omega)$, the space of square-integrable functions.

Power of Hilbert structure. Hilbert spaces combine the algebraic flexibility of vector spaces, the analytical control of norms, and the geometric richness of inner products.

Projection theorem

Theorem (Projection on a closed convex set)

Let H be a Hilbert space and $C \subset H$ a closed convex non-empty subset. For every $x \in H$, there exists a unique $p \in C$ such that

$$\|x - p\| = \inf_{c \in C} \|x - c\|.$$

The element p is the *orthogonal projection* of x onto C.

Characterization. p is the projection iff

$$\forall c \in C : \quad \langle x - p, c - p \rangle \leq 0.$$

Nonstandard proof. Pick a minimizing sequence (c_n), take an infinite index ν, apply standard part: $\mathrm{st}(c_\nu) \in C$ (by closure) achieves the infimum. Convexity + parallelogram law give uniqueness.

Riesz representation theorem

Theorem (Riesz)

Let H be a Hilbert space. For every continuous linear form $\varphi : H \to \mathbb{R}$, there exists a unique $v \in H$ such that

$$\forall x \in H : \quad \varphi(x) = \langle x, v \rangle.$$

Moreover, $\|\varphi\| = \|v\|$.

Reading. Every continuous linear form on H is represented by an inner product with a fixed vector. The dual space H^* is canonically isomorphic to H.

Proof strategy. Let $K = \ker \varphi$ (closed by continuity). If $K = H$, take $v = 0$. Else, pick $u \notin K$, project on K, and compute. The Riesz vector lives in the orthogonal complement of K.

Importance. Cornerstone of functional analysis, central in quantum mechanics (states as vectors, observables as operators).

The four pillars in summary

1. Transfer principle. First-order properties transfer between $\mathbb{R}$ and $\widetilde{\mathbb{R}}$. Logical bridge.

2. Standard part operator st**.** Projects limited hyper-reals to their classical counterparts. Ring morphism with kernel = infinitesimals.

3. Compactness via Robinson. Compact = every nonstandard element is nearly standard. Yields elegant proofs of Bolzano-Weierstrass, Heine, extreme values, intermediate values.

4. Hilbert spaces. Metric, Banach, pre-Hilbert, Hilbert — each extended nonstandardly. Projection theorem and Riesz representation as key results.

Unifying theme. Nonstandard analysis replaces ε-δ arguments with the intuitive notion of « infinitely close ». The grain program provides this approach with a *constructive geometric foundation*: infinitesimals are not formal abstractions but emerge from the grain mosaic.

Why this matters

Pedagogical.

- Proofs become significantly shorter and more intuitive.
- Direct conditions on infinite indices replace epsilon-management.
- Compactness theorems become almost transparent.

Conceptual.

- Infinitesimals are restored to mathematical respectability.
- The « two worlds » view ($\mathbb{R}$ vs. $\widetilde{\mathbb{R}}$) is structurally productive.
- The grain program gives nonstandard analysis a geometric anchor lacking in the Robinson-Loeb tradition.

Bridge to applications.

- Hilbert space framework — natural setting for quantum mechanics.
- Standard part — bridge between stratal regularization and classical limits.
- Compactness — controls convergence in nonstandard settings.

Open directions

Foundational.

- Full formalization of the transfer principle for second-order formulas.
- Characterization of formulas admitting transfer.

Analytical.

- Nonstandard measure theory on $\widetilde{\mathbb{R}}$.
- Distributions and Sobolev spaces.
- Spectral theory of operators on Hilbert spaces.

Topological.

- Nonstandard topology beyond metric spaces.
- Tychonoff theorem and product compactness.

Articulation with the grain program.

- Strata-aware analysis (operators preserving / crossing strata).
- Hilbert spaces over stratified scalars.
- Geometric interpretation of L^2 spaces in grain terms.

Thank you.

Questions?
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