# Functional and Multiplicative Stratification

From Cartesian anomaly to set-theoretic consequences

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# Outline

# This lecture

**Five topics, one structural thread.**

We develop a coherent line of results that ties together:

1. A geometric observation about Cartesian representation of functions.
2. Its application to the logarithm, producing the stratal metric.
3. The reciprocal construction (complex exponential) and its geometric forcing of $i$.
4. A unification: functional, multiplicative, and set-theoretic stratifications are facets of a single mechanism.
5. An application: regularization of divergent integrals through functional stratification.

**Common thread.** Multiplication is the structural operator that governs the strata hierarchy of the grain program. All five topics illustrate this in different guises.

# Setup

Consider a differentiable bijective function $f : [a, b] \to [c, d]$ and its inverse $g = f^{-1}$.

The arc of the graph $\Gamma_f = \{(x, f(x)) : x \in [a, b]\}$ has a geometric length $\ell$ that can be computed in two ways:

$$\ell_f = \int_a^b \sqrt{1 + f'(x)^2}\, dx = \int_c^d \sqrt{1 + g'(y)^2}\, dy.$$

**Algebraic conservation.** Both integrals give the same value, because the change of variable $y = f(x)$ introduces the Jacobian factor that compensates the deformation.

# The anomaly: moving to a new Cartesian frame

Now take $g = f^{-1}$ and draw it in a *new* Cartesian frame, with $X$ as the abscissa and $Y = g(X)$:

$$\ell_g^{\text{new}} = \int_c^d \sqrt{1 + g'(X)^2}\, dX.$$

**Claim.** In general, $\ell_g^{\text{new}} \neq \ell_f$.

**Counter-example.** For $f(x) = x^2$ on $[0, 1]$:

- $\ell_f = \int_0^1 \sqrt{1 + 4x^2}\, dx \approx 1.479$.
- $\ell_g^{\text{new}} = \int_0^1 \sqrt{1 + 1/(4y)}\, dy$ diverges at $y = 0$ (the curve $y = \sqrt{x}$ has a vertical tangent).

**Diagnosis.** The Cartesian frame imposes a uniform metric on both axes. The function $f$ deforms the codomain axis non-uniformly. Moving $f^{-1}$ to a new frame *forgets* the Jacobian factor, hence the discrepancy.

# The structural lesson

**Theorem (Functional stratification)**

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. The Cartesian representation $\Gamma_f = \{(x, f(x))\}$ induces on the codomain axis $Oy$ a *non-uniform stratification*:

- The uniform interval $[x, x + dx]$ maps to $[f(x), f(x) + f'(x)dx]$.
- Its length depends on $x$ through $|f'(x)|$.
- $f$ acts as a *local stratification factor* on the codomain.

**Reading.** The Cartesian plane is not a neutral representation: it imposes a uniform metric while functions induce local deformations. *Drawing a function is stratifying the vertical axis.*

**Historical connection.** This extends Bolzano's observations on images of the continuum: cardinality is preserved, but local measure is not.

# Setup: transporting a continuum

**Problem.** Transport a continuum $\mathcal{C}$ factorized as

$$f_1 = \left(1,\ 2^{2n_\infty},\ \frac{1}{2^{2n_\infty}}\right)$$

onto a reference continuum

$$f' = \left(1,\ 2^{n_\infty},\ \frac{1}{2^{n_\infty}}\right).$$

Same length (1), but different cardinality and different elementary thickness.

**Strategy.**

1. Homothety (factor $2^{n_\infty}$): same cardinality, thickness becomes $1/2^{n_\infty}$ (homogeneous in $\widetilde{\mathbb{R}}$).
2. Logarithmic contraction: brings the length back into $\mathbb{R}$, with heterogeneous thickness $\epsilon_h$.
3. Domain adjustment.

# The construction

**Step 1: Homothety.**

$$f_1' = \left(2^{n_\infty},\ 2^{2n_\infty},\ \frac{1}{2^{n_\infty}}\right).$$

**Step 2: Logarithmic contraction.**

$$f_1'' = \left(n_\infty \ln 2,\ 2^{2n_\infty},\ \epsilon_h\right),$$

where $\epsilon_h$ is the heterogeneous thickness induced by the logarithm.

**Step 3: Domain adjustment.**

- Reference continuum domain: $[0,1[$.
- Logarithmic image: $]-n_\infty \ln 2, 0]$ (in logarithmic coordinates).

**Result.** The continuum $\mathcal{C}$ is represented in $\mathcal{C}'$ under a *logarithmic scale* — equivalent to defining a *logarithmic metric* on the continuum.

# The stratal metric

## Definition (Stratal metric)

For positive reals $x, y \in \widetilde{\mathbb{R}}_*^+$, the *stratal distance* is

$$d_\sigma(x, y) := |\ln(y/x)|.$$

This is the metric induced by the logarithmic transport.

**Key properties.**

- **Invariance under multiplicative homothety:** $d_\sigma(\lambda x, \lambda y) = d_\sigma(x, y)$. Position has no privileged scale; only ratios matter.
- **Linearization:** $d_\sigma$ converts multiplicative stratification into additive distance.
- **Magnitude reduction:** the exponential invariant $2^{n_\infty}$ becomes the polynomial invariant $n_\infty \ln 2$.

**Universal signature.** This metric is the canonical coordinate of the program's stratification — emerging from the construction, not imposed externally.

# Setup: the reciprocal of $T = \log_2$

The stratal logarithm $T : \widetilde{\mathbb{R}}_*^+ \to \widetilde{\mathbb{R}}$ has a reciprocal: the *real stratal exponential*

$$E(\theta) := T^{-1}(\theta) = 2^\theta.$$

**Question.** Can $E$ extend to the unit circle in a coherent way?

**Circular compatibility constraints.**

- $\mathcal{E}(0) = +1$ (origin point).
- $\mathcal{E}(\pi) = -1$ (geometric antipode).
- $\mathcal{E}(2\pi) = +1$ (periodicity).
- $\mathcal{E}(\theta_1 + \theta_2) = \mathcal{E}(\theta_1) \cdot \mathcal{E}(\theta_2)$ (multiplicativity).

**Problem.** The real exponential $E(\theta) = 2^\theta$ fails: $E(\pi) = 2^\pi \neq -1$, not periodic.

# The forced emergence of $i$

Introduce a structural factor $\xi$: $\mathcal{E}(\theta) := 2^{\xi\theta}$.

**Theorem (Geometric emergence of $i$)**

The unique $\xi$ such that $\mathcal{E}(\theta) = 2^{\xi\theta}$ satisfies the circular constraints requires:

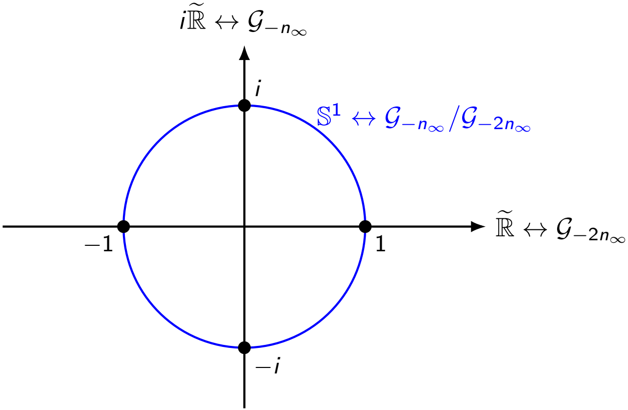
$$\boxed{\xi^2 = -1.}$$

This element is not in $\widetilde{\mathbb{R}}$: it belongs necessarily to an extension where $\sqrt{-1}$ exists. We call it $i$.

**Proof sketch.** Multiplicativity gives $\mathcal{E}(\pi) = \mathcal{E}(\pi/2)^2$. With $\mathcal{E}(\pi) = -1$, we get $\mathcal{E}(\pi/2)^2 = -1$.

**Key point.** $i$ is *not algebraically adjoined*. It is *geometrically forced* by the compatibility between the stratal exponential and the circular structure of the carrier circle.

# The stratal complex plane




**Three superposed objects on one geometric support:**

- Real axis $\widetilde{\mathbb{R}}$: fine deployment.
- Imaginary axis $i\widetilde{\mathbb{R}}$: coarse deployment.

# Euler's formula as consequence

## Stratal complex exponential

For $\theta \in \widetilde{\mathbb{R}}$:

$$\mathcal{E}(\theta) = 2^{i\theta/\ln 2} = e^{i\theta}.$$

**Properties.**

- $\mathcal{E}(0) = 1$, $\mathcal{E}(\pi) = -1$, $\mathcal{E}(2\pi) = 1$.
- $|\mathcal{E}(\theta)| = 1$ (values on the unit circle).
- $\mathcal{E}(\theta) = \cos\theta + i\sin\theta$.

**Euler's formula** $e^{i\pi} = -1$ becomes a *direct consequence* of the construction: it expresses that the half-turn on the carrier circle is the involution $-1$, an elementary geometric property.

# The structural question

We have established:

- *Functional stratification* (Cartesian anomaly): functions deform the codomain axis.
- *Multiplicative stratification* (Volume I): $x \cdot y$ for $x, y \in \mathcal{G}_{-n_\infty}$ lives in $\mathcal{G}_{-2n_\infty}$.

**Question.** Are these two mechanisms related?

**Answer.** They are the *same mechanism*, viewed from different angles. A third facet — set-theoretic — completes the unification.

# Local homothety induced by $f$

**Proposition (Local homothety)**

Let $f : \mathcal{C} \to \mathcal{C}'$ be differentiable. At each grain $G_k$ centered at $x_k$, the function $f$ acts locally as a *homothety of ratio* $\lambda_k := |f'(x_k)|$:

$$\epsilon_k = \lambda_k \cdot \epsilon_0.$$

**Key observation.** A homothety is a *multiplication*: multiplying positions by $\lambda_k$.

**Consequence.** By the multiplicative stratification of Volume I, this multiplication makes the grain *exit its stratum* toward the finer level. The resolution $-2n_\infty$ emerges naturally as the level where the heterogeneity of $f$ is absorbed into a homogeneous mosaic.

# Functional = multiplicative stratification

**Theorem (Equivalence)**

The functional stratification induced by $f$ is *identical* to the local multiplicative stratification of Volume I:

- At each grain $G_k$, $f$ acts by homothety of ratio $\lambda_k$.
- This homothety is a multiplication, which exits the stratum $\mathcal{G}_{-n_\infty}$ to enter $\mathcal{G}_{-2n_\infty}$.
- *$f$ lifts each grain by one stratum*, explaining the natural emergence of resolution $-2n_\infty$.

**Conceptual reading.**

- Algebraic view (Vol. I): $x \cdot y$ stratifies.
- Geometric view (function): $f$ deforms locally.

*Both describe the same operation: exit from stratum by multiplicative operation.*

# Three counts of the image

For an image continuum $\mathcal{C}'$:

**Positional count at $-n_\infty$:**

$$|\mathcal{C}'|_{/\mathcal{G}_{-n_\infty}} = 2^{n_\infty} \in \widetilde{\mathbb{N}}_1.$$

**Positional count at $-2n_\infty$:**

$$|\mathcal{C}'|_{/\mathcal{G}_{-2n_\infty}} = 2^{2n_\infty} \in \widetilde{\mathbb{N}}_2.$$

**Combinatorial count (configurations of grains seen at $-2n_\infty$):**

$$|\mathcal{B}_{-n_\infty}|^{\text{seen at } \mathcal{B}_{-2n_\infty}} = 2^{n_\infty} \cdot 2^{2n_\infty} = 2^{3n_\infty} \in \widetilde{\mathbb{N}}_3.$$

**Three distinct strata.** The combinatorial count goes one stratum higher than positional counts.

# The unification theorem

**Theorem (Unification)**

Three mechanisms are *three facets of one structural phenomenon*:

1. **Functional stratification.** $f$ stratifies the codomain heterogeneously at $-n_\infty$, homogeneously at $-2n_\infty$.
2. **Multiplicative stratification.** $x \cdot y$ exits the stratum $\mathcal{G}_{-kn_\infty}$ toward $\mathcal{G}_{-(k+1)n_\infty}$.
3. **Stratal cardinal exponentiation.** The positional count $2^{2n_\infty}$ becomes the combinatorial count $2^{3n_\infty}$.

These are respectively the *geometric*, *algebraic*, and *set-theoretic* expressions of *exit from stratum by multiplicative operation*.

**Conclusion.** *The product is the unique mechanism governing the strata hierarchy of the grain program.*

# Setup: improper integrals as measure problems

Consider an improper integral

$$I = \int_a^{+\infty} f(x)\, dx.$$

**Implicit assumption.** We use the Euclidean Lebesgue measure $dx$ on the integration axis. This is a *convention*, not a necessity.

**Idea.** Use a different measure on the axis, induced by a *stratification function* $g$:

$$d\mu_g(x) = |g'(x)|\, dx.$$

This is formally a *Lebesgue-Stieltjes integral* with integrator $\alpha = g$.

# Functional stratification regularization

## Stratified integral

$$I_g(f) := \int_a^{+\infty} f(x)\, d\mu_g(x) = \int_a^{+\infty} f(x)|g'(x)|\, dx.$$

**Principle.** For any divergent $\int f$, there exists a class of *regularizing stratifications* $g$ such that $I_g(f)$ converges.

**Example:** $\int_1^{+\infty} dx/x$ **(diverges).**
Logarithmic stratification $g = \ln$: $d\mu_{\ln}(x) = dx/x$.

$$I_{\ln}(1/x) = \int_1^{+\infty} \frac{1}{x} \cdot \frac{1}{x}\, dx = \int_1^{+\infty} \frac{dx}{x^2} = 1.$$

**Note.** Different $g$ give different regularized values. *Canonical choices* are needed for the operation to be meaningful.

# Why the logarithm is canonical

The logarithmic stratification $g = \ln$ is privileged:

- **Invariant under multiplicative homothety:** $g(\lambda x) - g(\lambda y) = g(x) - g(y)$. The measure $dx/x$ is invariant under multiplicative scaling.
- **Corresponds to the stratal metric:** it is the construction we have seen in Section 2.
- **Linearizes multiplicative structures:** converts products into sums.

**Reading.** For scale-free phenomena (no characteristic scale), multiplicative invariance forces the logarithmic measure as the *unique canonical choice*.

# Relation to Stieltjes integration

**Formal kinship.** The stratified integral $I_g(f)$ is, formally, a Lebesgue-Stieltjes integral. This classical framework (since Riemann-Stieltjes 1894, extended by Lebesgue) defines $\int f\,dg = \int fg'\,dx$ for $g$ of bounded variation.

**What our approach adds.**

- Classical Stieltjes: the integrator $g$ is a *free choice* dictated by the problem (e.g., probability distribution).
- Stratal approach: the integrator $g$ is *canonical*, imposed by the structure of the grain program (multiplicative invariance, stratal metric).

**The formulation is Stieltjes; the interpretation is the grain program's.**

# Two complementary mechanisms

**Mechanism 1: weighted measure (Stieltjes).**

- Effective for divergences *at infinity*.
- Choose $g$ such that $|g'(x)| \to 0$ fast enough as $x \to +\infty$.

**Mechanism 2: stratal cutoff (Volume III).**

- Effective for divergences *at a point*.
- Replace $\int_0^1$ by $\widetilde{\int}_{\Delta x}^{1}$ with $\Delta x = 2\pi/2^{n_\infty}$.
- The cutoff $\Delta x$ is *structural*, not arbitrary.

**Common spirit.** Both modify the integration procedure *canonically* to handle divergences. They are distinct but complementary mechanisms.

**Result of Volume III:** $\widetilde{\int}_{\Delta x}^{1} dx/x \sim n_\infty \ln 2 \in \widetilde{\mathbb{R}}_{|1}$.

# The unifying theme

**Multiplication as structural operator.**

- *Geometrically* (functions): homothety.
- *Algebraically* (Vol. I): exit from stratum.
- *Set-theoretically* (cardinals): stratal exponentiation.
- *Topologically* (metric): logarithmic linearization.
- *Analytically* (integrals): canonical regularization measure.

**The five topics are facets of one structural insight:** the hierarchy of strata in the grain program is governed by multiplication, viewed under different mathematical lenses.

**Implications.**

- The Cartesian plane carries hidden stratifications.
- The logarithmic metric is the canonical coordinate of these stratifications.
- $i$ emerges geometrically from circular compatibility.
- Divergent integrals are regularizable through canonical stratifications.

# Open directions

**Geometry.**

- Other stratification functions (powers, exponentials, trigonometric).
- Higher-dimensional stratifications.

**Algebra.**

- Other stratal operations (addition, division, composition).
- Structure of cardinalities mixing different levels.

**Set theory.**

- Hierarchy of stratal cardinals: $2^{n_\infty}, 2^{2n_\infty}, 2^{3n_\infty}, \ldots$
- General formula for combinatorial counts on $k$ levels.

**Analysis.**

- Combining weighted measure and stratal cutoff.
- Stratal interpretation of measure-theoretic regularization.

# Thank you.

Questions?

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*Grain Program*

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