Achilles, the Tortoise and the Grain

Zeno's Paradox and the Level of Observation

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The aim of this talk

- For twenty-five centuries, Zeno has claimed that Achilles never catches the tortoise.
- The classical answers tell us where and when Achilles catches it; they say little about how.
- We propose another reading, based on a simple idea: we only ever see at a given level of observation, with a "point" that has a size.
- Thesis: Zeno's "never" is not a miscalculation but a choice of viewpoint. With a fixed viewpoint, Achilles catches the tortoise.

Outline

1. Zeno and his time
2. The paradox of Achilles
3. The classical answers
4. The grain and the level of observation
5. The race seen through the grain
6. Two viewpoints on the same race
7. Conclusion

1. Zeno and his time

Zeno of Elea (c. 490 BC)

- Born in Elea, a Greek colony in southern Italy.
- Disciple of Parmenides: being is one, continuous and motionless; change is mere appearance.
- In the *Parmenides*, Plato shows him in Athens at about forty, facing the young Socrates.
- His book defended his master *by reductio ad absurdum*: the opposing thesis, plurality and motion, leads to even worse contradictions.
- Aristotle regarded him as the inventor of dialectic (according to Diogenes Laertius).

A method

Do not prove your own thesis; show that your opponent's thesis contradicts itself.

The sources

The book is lost. We read Zeno through Plato, Aristotle (*Physics*, VI) and Simplicius.

The four arguments against motion

1. The dichotomy

To reach the end, one must first reach the midpoint, then the midpoint of what remains… One never arrives.

2. Achilles and the tortoise

The fastest never catches the slowest.

3. The arrow

At every instant, the arrow occupies a space equal to itself: it is at rest at every instant.

4. The stadium

Two rows of bodies passing each other: half a time would equal its double.

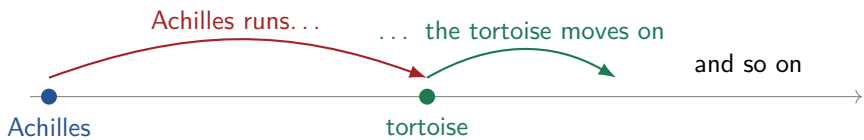
Aristotle himself notes it: Achilles is a variant of the dichotomy. In both cases, the remaining distance is divided without end.

2. The paradox of Achilles

The argument, as reported by Aristotle

In substance (*Physics*, VI, 9)

The slowest runner will never be overtaken by the fastest. For the pursuer must first reach the point from which the pursued started, so that the slower one always holds some lead.



Each time Achilles reaches the place where the tortoise was, it is no longer there.

The logical structure of the paradox

P1 To catch the tortoise, Achilles must first reach each of the points it has just left.

P2 These points are infinitely many: at each step, a new one remains.

P3 One cannot complete an infinite sequence of steps.

C Therefore Achilles never catches the tortoise.

Where should the criticism bear?

P1 and P2 are facts of geometry. Everything turns on **P3**, and on the meaning we give to "infinite" and "complete".

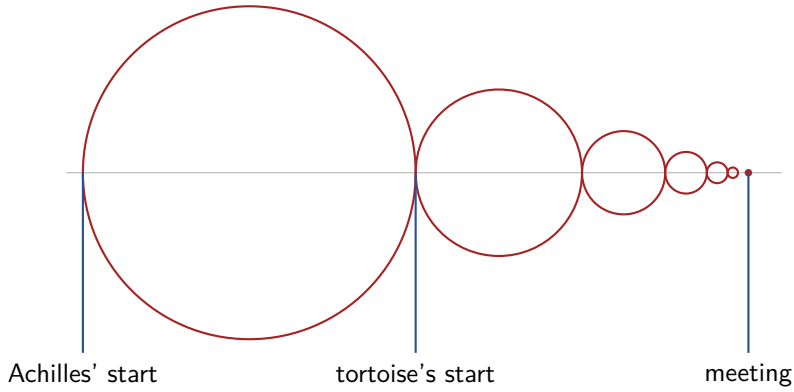
Putting in numbers

Achilles runs twice as fast. The tortoise starts with a lead of 1.

Step	Achilles	Tortoise	Gap
0	0	1	1
1	1	1.5	$1/2$
2	1.5	1.75	$1/4$
3	1.75	1.875	$1/8$
4	1.875	1.9375	$1/16$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
x	$2 - \frac{2}{2^x}$	$2 - \frac{1}{2^x}$	$\frac{1}{2^x}$

- The gap is halved at each step.
- It is never zero: $1/2^x > 0$.
- The positions approach 2 without reaching it.
- The point 2 is the meeting point: this is where we "know" that Achilles reaches the tortoise.

A picture: the grains separating Achilles from the tortoise



Each circle is a step. They shrink endlessly towards the meeting point. Zeno's question: how can one cross *all* the circles?

3. The classical answers

Aristotle: the potential infinite

- **First answer** (*Physics*, VI): time is divisible just as length is. To infinitely many segments correspond infinitely many intervals of time, ever shorter.
- **Second answer** (*Physics*, VIII): a line is not made of points. The points of division exist only potentially, as long as one does not stop at them.
- So Achilles does not run through an actual infinity of steps; he runs a single distance, which thought can divide without end.

Key point

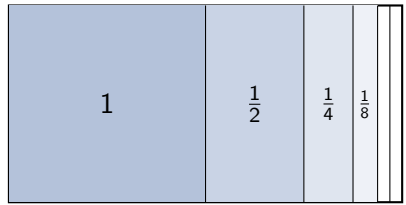
Aristotle shifts the problem: the infinite is not in the race, it is in the division we impose on it.

The sum of an infinite series

- Archimedes already knows how to sum a geometric progression.
- Grégoire de Saint-Vincent (1647) applies it explicitly to Achilles.
- With Cauchy (1821) and the notion of limit:

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots = 2.$$

- The durations of the steps also form a convergent series: Achilles catches the tortoise in a finite time, at the point 2.



the sum fills the 2×1 rectangle

What the limit says, and what it does not say

What it says

- Where: at the point 2.
- When: after a finite time.
- The infinite sum has a precise value.

What it does not say

- How one completes infinitely many steps.
- What happens "at the end", since there is no last step.

- **Bergson** (1889): we confuse motion, an indivisible act, with the trajectory, divisible at will.
- **Thomson** (1954): the lamp switched on at 0, off at $\frac{1}{2}$ minute, on again at $\frac{3}{4}$, and so on. Is it on or off at 1 minute? The series converges, but the question remains unanswered. This is the problem of supertasks.

The blind spot of the classical answers

A shared presupposition

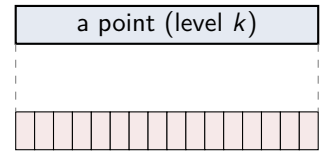
All these answers assume that there exists a point, without thickness, where Achilles and the tortoise meet: the point 2.

- Euclid: "a point is that which has no part".
- Yet Aristotle: a line is not made of points.
- Question: what if the meeting point were not a point without thickness, but a grain? What if "catching up" always meant "catching up *to within a certain grain*"?

4. The grain and the level of observation

An intuition: the pixel

- On a screen, the smallest visible thing is the pixel.
- What is smaller than a pixel is invisible; what contains millions of them looks continuous.
- Under a magnifying glass, a pixel becomes a whole picture.
- What counts as "a point" therefore depends on the scale of observation.



The theory of the grain

This is not a flaw of the representation: it is the very structure of the continuum. Every "point" we see is a grain.

Numbers larger than all numbers

- In 1934, **Skolem** shows that one can add to the usual integers $0, 1, 2, \ldots$ some unlimited integers, larger than all the usual ones, without contradiction.
- They obey the same rules of arithmetic (*transfer*).
- We fix one such unlimited integer, written ν: it is "the number of the integers", their numerosity.
- Its inverse $1/\nu$, and even more so $2^{-\nu}$, is infinitesimal: smaller than $1/n$ for every usual integer n.

For the philosopher

The infinitely small, banished by Cauchy and Weierstrass, returns as a *genuine number*, which can be divided, compared and counted.

The absolute point is a grain

- Let us surround a point a with smaller and smaller intervals: $[a-1, a+1]$, $[a-\frac{1}{2}, a+\frac{1}{2}]$, $[a-\frac{1}{3}, a+\frac{1}{3}]\ldots$
- On the usual line, only a remains in the end: it is an absolute point, of zero thickness.
- With infinitesimals, a whole zone remains: $a+2^{-\nu}$, $a+2^{-2\nu}\ldots$ This is the monad of a.



Thesis

What the usual line sees as a point without thickness is, seen more closely, a grain that can be divided without end.

The level of observation

Definition (Level of observation)

At level of observation k, the unit of vision is a grain of thickness

$$\delta_k = 2^{-k\nu},$$

called the resolution point. An object X contains $\#_k(X) = |X|/\delta_k$ resolution points at this level.

- Level 0: the resolution point is the unit; this is the human scale.
- Level 1: grains of thickness $2^{-\nu}$, already infinitesimal.
- Going from level k to level $k+1$ means using a magnifying glass of power 2^{ν}.

Punctual, limited, at infinity: relative statuses

Definition (Relative status)

At level k, an object X is

- punctual if it fits within one resolution point: $\#_k(X) \leq 1$;
- limited if $\#_k(X)$ is a usual number;
- at infinity if $\#_k(X)$ is unlimited.

Proposition (Relativity)

$\#_{k+1}(X) = 2^{\nu}\,\#_k(X)$. An object that is limited at level k is at infinity at level $k+1$.

Key point

"Punctual" is not a property of a thing, but of a thing at a level of observation. Every statement about a point must say at which grain it speaks.

5. The race seen through the grain

First finding: Zeno is right on one point

With a lead x_0, at rank x:

$$\text{gap} \quad \Delta_x = \frac{x_0}{2^x}.$$

- This formula holds for all ranks, usual or unlimited.
- The gap is never zero, even at rank ν, even at rank ν^2.

In all honesty

Adding unlimited numbers does not make the paradox disappear. At no rank is Achilles *exactly* on the tortoise.

Consequence

"Achilles catches the tortoise" cannot mean "the gap is zero". It means: the gap is no longer discernible. So we must say at which level of observation.

Second finding: seen from afar, Achilles has reached the tortoise

Proposition

At every unlimited rank, Achilles and the tortoise both lie in the monad of the meeting point: their gap is infinitesimal.

- This is the classical answer, that of the limit: to the naked eye, and even with every usual magnifying glass, Achilles is on the tortoise.
- But the monad is not a point: it is a grain. What happens inside?

The catching-up rank at a given level

Definition

Achilles has caught the tortoise at level k when the gap is punctual there: $\Delta_x \leq \delta_k$.

Theorem (Catching-up rank)

Let ℓ be the smallest integer such that $2^\ell \geq x_0$. At level $k \geq 1$, Achilles catches the tortoise exactly at rank

$$r_k = k\nu + \ell.$$

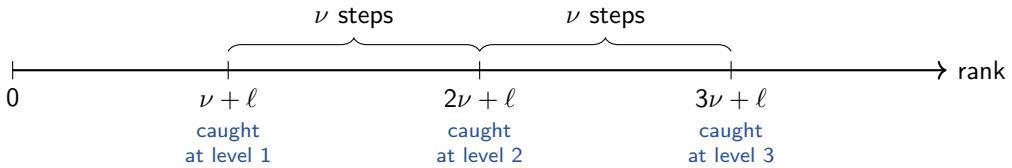
Proof. $\Delta_x \leq 2^{-k\nu}$ is equivalent to $2^{x-k\nu} \geq x_0$, that is, $x - k\nu \geq \ell$.

Example: lead $x_0 = 1$, hence $\ell = 0$

At level 1, caught at rank ν; at level 2, at rank 2ν; at level 3, at rank 3ν.

Each level of observation adds ν steps

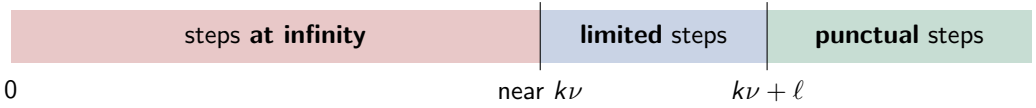
$$r_{k+1} - r_k = \nu.$$



What this result means

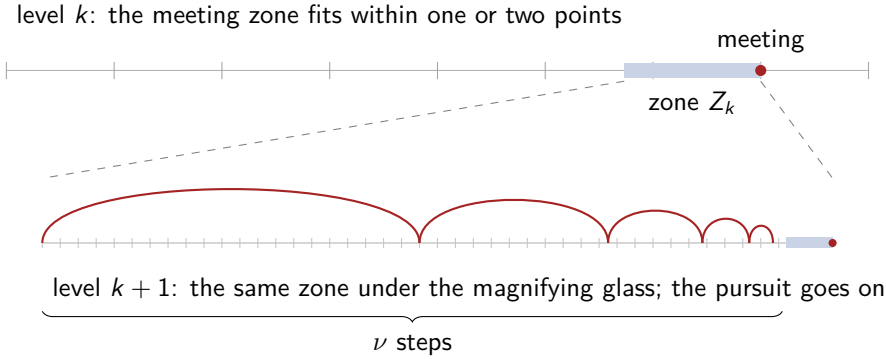
Zeno's "never" becomes "not before rank $k\nu + \ell$". This rank exists at every level. What does not exist is a level of zero thickness, which would make the meeting point an absolute point.

The three regimes of the pursuit



- **At infinity**: for an unlimited number of steps, each one contains infinitely many resolution points. All the "human" steps are here.
- **Limited**: around rank $k\nu$, the steps have the size of what one sees at this level.
- **Punctual**: from rank $k\nu + \ell$ on, they fit within one resolution point. The race is completed at this level.

The meeting point is a grain



Thesis

The zone where Achilles "has reached" the tortoise at level k is, at level $k+1$, a stretch in which the pursuit still lasts ν steps. No level turns it into an absolute point.

6. Two viewpoints on the same race

The race looks the same at every scale

- Magnify the race twice around the meeting point: step x becomes exactly step $x - 1$.
- Magnify it 2^ν times: step x becomes step $x - \nu$, and the meeting zone of level $k + 1$ becomes that of level k.
- The race is self-similar, like a rocky coastline or a Romanesco broccoli.

Hence the question

When we magnify, what does the observer do? Does he keep his resolution point, or does he magnify it along with the picture?

Two readings of the dilation

(A) The viewpoint follows the race

The observer enlarges his resolution point with each step. Each step then has the *same* number of points as the previous one: the race repeats itself, identical, without end.

This is Zeno's viewpoint.

(B) The viewpoint stays fixed

The observer keeps his resolution point. The steps contain fewer and fewer points, then fit within a single one.

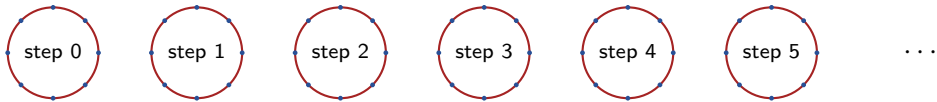
The race is completed at rank $k\nu + \ell$.

Proposition

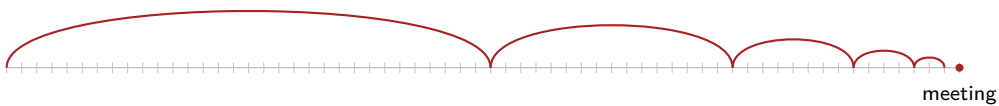
Reading (A) at level $k+1$ coincides with reading (B) at level k: the two viewpoints differ by one level of observation.

The two viewpoints in a picture

(A) the viewpoint follows the step



(B) the viewpoint stays fixed



In (A), there are always eight resolution points (in blue) per step: nothing ever changes. In (B), the resolution points (grey ticks) keep their size: the last steps fit within a single one.

Proper part and partial part

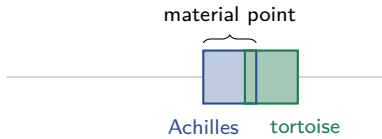
- The same collection can be viewed in two ways:
  - as a partial part: taken within a whole, counted at the grain of that whole;
  - as a proper part: taken for itself, each element renumbered as a unit.
- **Zeno** renumbers each step as a unit: the steps form a proper part, with as many elements as the integers, which is never completed.
- **The race** takes place on a segment, counted at a fixed grain: the steps form a partial part, which is completed.

Viewpoint	Resolution point	The race
proper part (Zeno)	shrinks with the step	is never completed
partial part, level k	fixed, $2^{-k\nu}$	is completed at rank $k\nu + \ell$

7. Conclusion

The material point: a fixed level of observation

- In mechanics, Achilles and the tortoise are material points: bodies whose size is neglected, but which do have one.
- The material point is exactly a level of observation taken as fixed: its thickness does not shrink with the race.
- This is reading (B). As soon as the gap fits within the material point, the two runners can no longer be told apart.



Conclusion

The level of observation, taken as the material point, and therefore fixed, shows that Achilles catches the tortoise, and it tells us at which rank: $k\nu + \ell$ at level k.

What we grant to each

- **To Zeno**: the gap is never zero. If one follows the race with a viewpoint that narrows along with it, the race is never completed. His "never" is an exact description of that viewpoint.
- **To Aristotle**: the points of division exist only potentially. At a given level, there is only a determinate number of discernible steps, $k\nu + \ell$.
- **To Cauchy**: seen from afar, at every usual resolution, Achilles reaches the tortoise at the limit point.
- **What the grain adds**: the meeting point is a grain, and "catching up" is relative to a level of observation. The paradox arises when one changes level without saying so.

Topics for discussion

1. **The arrow.** At a given level of observation, does an "instant" have a duration? Is the arrow at rest within a grain of time?
2. **Thomson's lamp.** At level k, are the switchings beyond rank $k\nu$ still discernible? Does the question "on or off?" make sense at a given level?
3. **The material point.** Is it a physicist's idealisation, or the very form of all observation?
4. **Bergson.** Is Bergson's "indivisible" motion simply motion seen at a fixed level?

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