

The Cartesian anomaly and the field

The equivalence principle read as a grain of the momentum axis

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What this talk is about

The question

A body in a uniform field gains momentum at a constant rate: $dq = C dt$. In what sense is this “the same” as an accelerated frame, and what does the **grain** (stratification) of the momentum axis tell us about it?

Three threads, woven step by step:

- 1 **History**: where the equivalence principle comes from.
- 2 **Grain**: the Cartesian anomaly (a function stratifies its target axis) applied to the momentum q .
- 3 **Status**: what is proved, what is measured, what is postulated — and a conjecture about time.

Outline

- 1 Where the principle comes from
- 2 The tool: grain and Cartesian anomaly
- 3 Step by step: the uniform field
- 4 The reverse direction: the grain of time
- 5 Connection versus curvature
- 6 Status: why a “principle”?
- 7 A conjecture about time

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Galileo (1590s–1638): all bodies fall alike

- Against Aristotle (heavy bodies fall faster), Galileo argues that, air resistance aside, **all bodies fall with the same acceleration**.
- His thought experiment: tie a heavy and a light stone together. Aristotle predicts both “the pair falls faster” (heavier) and “slower” (the light one brakes) — a contradiction.
- With inclined planes he finds the **law of odd numbers**: distances covered in successive equal times are in the ratio $1 : 3 : 5 : 7 : \dots$ (*Two New Sciences*, 1638).

Keep in mind

The odd-numbers law will reappear later, derived from the *homogeneous grain* of q .

Newton (1687): two masses that happen to agree

In Newton's mechanics mass plays **two different roles**:

$$\underbrace{F = m_{\text{inertial}} a}_{\text{resistance to acceleration}}$$

$$\underbrace{F = m_{\text{grav}} g}_{\text{coupling to gravity}}$$

- Nothing in the theory forces $m_{\text{inertial}} = m_{\text{grav}}$.
- If they are equal, then $a = \frac{m_{\text{grav}}}{m_{\text{inertial}}} g = g$ for every body: Galileo's universality of free fall.
- Newton tested it with pendulums of different materials (gold, silver, lead, glass, sand, wood...) to about one part in 10^3 .

A coincidence without an explanation

For Newton, the equality is an experimental fact, not a consequence.

Eötvös (1889–1908) and modern tests

- Loránd Eötvös, torsion balance: compares the pull of gravity and the centrifugal effect of Earth's rotation on different materials. Precision about 10^{-8} to 10^{-9} .
- Twentieth century: Dicke, Braginsky, lunar laser ranging (Earth and Moon falling toward the Sun).
- **MICROSCOPE** satellite (final results 2022): equality verified to about 10^{-15} .

What is being tested

Always the same thing: $m_{\text{grav}}/m_{\text{inertial}}$ is the same for all bodies. This is the *weak* equivalence principle.

Einstein (1907): “the happiest thought of my life”

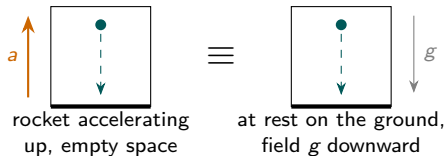
- 1907, Bern patent office: *a person falling freely from a roof does not feel their own weight*. Objects released next to them float.
- Einstein turns Newton’s coincidence into a **principle**: locally, a uniform gravitational field and a uniformly accelerated frame are **physically indistinguishable**.
- He extends it from mechanics to *all* physics (light, clocks, electromagnetism). This is the *Einstein* equivalence principle.

First predictions (1907–1911)

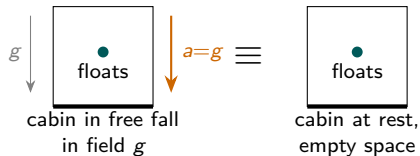
Clocks run at different rates at different heights (gravitational redshift); light is bent by gravity. Both follow from the principle *before* general relativity (1915).

The elevator: two faces of one principle

Face 1: acceleration creates a field



Face 2: free fall erases the field



One formula for both (upward axis)

A free body seen from a cabin of acceleration a , in a field g : $\ddot{y}_{\text{rel}} = g - a$. Face 1:
 $g = 0 \Rightarrow \ddot{y}_{\text{rel}} = -a$. Face 2: $a = g \Rightarrow \ddot{y}_{\text{rel}} = 0$.

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A function stratifies its target axis

Functional stratification (Theorem 62)

Let f be differentiable. The uniform interval $[x, x + dx]$ is sent to $[f(x), f(x) + f'(x) dx]$. The target axis inherits a local **grain**, the stratification factor $|f'(x)|$.

- **Cartesian anomaly**: the grain is not constant. Drawing a function stratifies the vertical axis unevenly.
- It disappears exactly when $|f'|$ is constant (isometry, factor ± 1).
- **Warning example** (Prop. 76): $f(x) = x^2$ has $f'(0) = 0$; its inverse $\sqrt{y}$ has a vertical tangent at 0. A **zero factor one way is an infinite factor the other way**. (We will need this at the end.)

Along a trajectory $t \mapsto y(t)$:

- $q = m\dot{y}$ — momentum (the “dynamical quantity”);
- $C = m\ddot{y}$ — force;
- grain of q *against* an axis u : the factor dq/du .

The key point

A grain is always read *against* some axis. The same motion can have a homogeneous grain against one axis and a heterogeneous grain against another.

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Step 1 — Uniform acceleration

Definition

A system is **uniformly accelerated** if the force is constant:

$$dq = C dt, \quad C = \text{const}, \quad q(t) = q_0 + C t.$$

Equivalence with a constant field

A constant force gives every body the same acceleration $a = C/m$ *provided* C is proportional to m . It is then locally indistinguishable from a **constant vector field** $g = a$ acting identically at every point of space.

Exact for a uniform field; only *local* for real gravity (see tides).

Step 2 — Grain of q against time: homogeneous

Proposition

Against time, $q(t) = q_0 + Ct$ is affine:

$$\frac{dq}{dt} = C \quad (\text{constant}).$$

No anomaly: the temporal grain of q is homogeneous.

Each equal time step produces an equal momentum step: the ticks on the q axis are evenly spaced (panel a).

Step 3 — Grain of q against space: heterogeneous

Proposition

By the chain rule along the trajectory,

$$\frac{dq}{dy} = \frac{dq/dt}{dy/dt} = \frac{C}{\dot{y}} = \frac{mC}{q}.$$

Integrating $q dq = mC dy$ from rest:

$$q(y) = \sqrt{2mC(y - y_0)} \propto \sqrt{y - y_0}.$$

The grain diverges at the turning point ($q = 0$) and thins out downstream (panel b).

Step 4 — Reconciling the two readings

“ q drifts at a constant rate” and “ q is deformed” are **both true**:

$$\frac{dq}{dy} = \underbrace{\frac{dq}{dt}}_{= C \text{ (the field)}} \cdot \underbrace{\frac{dt}{dy}}_{= 1/\dot{y} \text{ (kinematics)}} = \frac{C}{\dot{y}}.$$

- The **homogeneity lives in the field**: the constant factor C in $dt \mapsto C dt$.
- The heterogeneity against space does *not* come from the field, but from $1/\dot{y}$: the trajectory samples space faster and faster.

The dividing line

$$\underbrace{dt \mapsto C dt}_{\text{homogeneous}} \Leftrightarrow \Gamma \text{ removable}; \quad \underbrace{dt \mapsto C(y) dt}_{\text{inhomogeneous}} \Leftrightarrow R \text{ irreducible (tides).}$$

Step 5 — The discrete grain of q

Hyperinteger n_∞ , integer $k \geq 1$, $\varepsilon = 2^{-kn_\infty}$. Normalised fall from rest:

$$\tau = t/T, \quad Q = q/(CT), \quad X = x/L: \quad Q(\tau) = \tau, \quad X(\tau) = \tau^2 = Q^2.$$

Time and momentum have **weight 1**; space has **weight 2**.

Theorem (homogeneous grain of q)

Cut time into uniform grains $\tau_m = m\varepsilon$. During the m -th grain of time, momentum runs through

$$H_m = [Q(\tau_m), Q(\tau_{m+1})] = [m\varepsilon, (m+1)\varepsilon]:$$

exactly **one** grain of the same resolution, for every m .

In physical units: a time grain of thickness $T\varepsilon$ gives a momentum grain of thickness $CT\varepsilon$ — the homogeneous deformation $dt \mapsto C dt$.

Step 6 — The grain of q generates Galileo's odd numbers

Since $X = Q^2$, the m -th space grain is the square of H_m :

$$|G_m| = Q(\tau_{m+1})^2 - Q(\tau_m)^2 = (Q(\tau_m) + Q(\tau_{m+1})) |H_m| = (2m + 1) \varepsilon^2.$$

- $2m + 1 = m + (m + 1)$: the sum of the cumulated momentum grains at the two ends of the time grain.
- Distances in successive equal times: $1 : 3 : 5 : 7 : \dots$ — Galileo's law of odd numbers.
- Consecutive odd numbers differ by $2 = 2 \times 1$: the odd-numbers law is the trace, at weight 2, of the homogeneous grain of weight 1.

Reading

Galileo's empirical law (1638) is the spatial shadow of the homogeneous grain of momentum.

Step 7 — Triple equivalence

Proposition 1

The following are equivalent:

- (i) **Homogeneous stratal deformation:** *for every body and every place, each time grain produces a momentum grain of thickness $CT\varepsilon$, with $C = mg$ and g the same for all;*
- (ii) **Homogeneous field:** *a constant vector field g acts identically at every point of space;*
- (iii) **Uniformly accelerated frame:** *free bodies described from a frame accelerating by $-g$ relative to an inertial frame.*

Proof sketch. (i) $\Leftrightarrow$ (ii): same grain for every body and place is the definition of a homogeneous field. (ii) $\Leftrightarrow$ (iii): with $y = y' + \frac{1}{2}gt^2$, one gets $q = q' + mgt$. A free body (q' constant) shows exactly grains of thickness $mgT\varepsilon$. No measurement inside the frame separates the two descriptions.

Step 8 — Erasure in the free-fall frame

In the frame accelerating with a : $q' = q - Ct$, $x' = x - \frac{1}{2}at^2$.

	lab (field)	free fall
momentum grains per time grain	1 (weight 1, homogeneous)	0
fine space grains in G_m	$2m + 1$ (weight 2, odd numbers)	1 or 0 ($p = 1$)

- Starting from rest, the body is **at rest** in this frame: all junctions coincide.
- With an initial velocity v_0 : $x' = v_0 t$, each time grain gives one uniform space grain — **no stratification left**.

Step 8 (figure) — The field is erased

Panel (c): subtracting Ct flattens q to q_0 — inertia recovered. Panel (d): equally spaced points in y' — uniform motion, trivial grain.

Step 9 — A global, not local, transformation

- Passing to free fall is **global**: the same at every point and every instant.
- It only erases a *homogeneous* field.
- If $C = C(y)$ depended on place, two neighbouring bodies would receive different grains, and no single global transformation would set both to zero.

What general relativity does

Local free-fall frames, one per region, glued together by a rule — the connection Γ . The obstruction to gluing them into one global inertial frame is the curvature R .

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Momentum as a clock

Proposition 2 (Homogeneous inverse)

In a uniform field ($C \neq 0$), $q(t) = q_0 + Ct$ is strictly monotone, hence invertible:

$$t(q) = \frac{q - q_0}{C}, \quad \frac{dt}{dq} = \frac{1}{C} \text{ (constant).}$$

Each momentum grain gives back exactly one time grain.

- Momentum is a good **clock** in a uniform field.
- But this is a *relabelling*, not a deformation: Newtonian time is absolute.
- Not circular to note, circular to use: the inverses agree *because* the laws of motion already agree. A bijection creates no information.

Limits

In free fall q' is constant: not invertible, so q stops being a clock. For an oscillator, $q(t)$ is not monotone and the inverse is multivalued.

The real deformation of time: gravitational redshift

Proposition 3 (Clocks in an accelerated frame)

Two clocks at rest, separated by a height h along the acceleration g :

$$d\tau_{top} = \left(1 + \frac{gh}{c^2}\right) d\tau_{bottom} \quad (\text{first order in } gh/c^2).$$

Sketch. Light takes $\Delta t \approx h/c$ to climb. Meanwhile the receiver gains $\Delta v = gh/c$ and recedes. First-order Doppler: $f_{\text{rec}} = f_{\text{em}}(1 - gh/c^2)$.

By the equivalence principle

The same holds in a uniform field; in a weak field, $d\tau = (1 + \Phi/c^2) dt$.

This one is **not** circular: the input is the principle plus light propagation, the output is a new prediction. The factor contains c^2 : we have left Newton.

- **Pound–Rebka** (1960), Harvard tower, $h = 22.5$ m: $gh/c^2 \approx 2.5 \times 10^{-15}$.
- **Optical clocks** (2010): detected for a height difference of 33 cm, $gh/c^2 \approx 3.6 \times 10^{-17}$.
- **GPS**: gravitational effect makes satellite clocks gain about $45 \mu\text{s}$ per day; it must be corrected.

Grain reading

Cut the bottom clock's proper time into uniform grains. Seen from above, each is stretched by the *same* factor $1 + gh/c^2$: a homogeneous stratal deformation of the time grain, fixed by height.

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The uniform field is a connection term

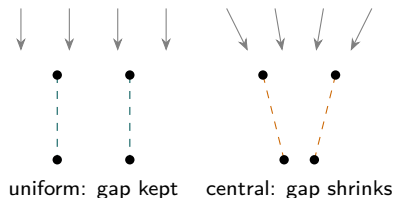
- The heterogeneous grain $dq/dy = mC/q$ is a **frame artefact**: in free fall, q is constant and the anomaly disappears.
- In affine language, a uniform field is a pure **connection term** Γ , removable by one global change of frame — exactly like Coriolis and centrifugal forces.
- In this precise sense, a uniform field is “fake” gravity: a Cartesian anomaly of the frame, not a property of space.

The honest boundary: tides and curvature

Real gravity points toward a centre and weakens with distance. Two bodies released side by side **converge**. This relative drift (tides) cannot be removed by any change of frame.

$$\frac{D^2 \xi^i}{d\tau^2} = - R^i_{jkl} u^j \xi^k u^l$$

ξ : separation of neighbouring geodesics; u : four-velocity.



Summary

Uniform part = connection Γ (removable). Non-uniform part = curvature R (irreducible). The signature of real gravity is the **failure of constancy**.

Mass as the hinge

Why can this field be absorbed into geometry at all?

- Because $C = ma$ with a **the same for all bodies**: universality of free fall.
- Contrast: an electric field gives $q_{\text{el}}E$, and q_{el}/m varies from body to body — impossible to geometrize.
- The **gravitational charge is the mass**; the equality of inertial and gravitational mass is exactly what makes the operation legitimate.

Link with the generalised dynamics

In the thread “ q may be any charge”, gravity is the limit case where the charge *is* the inertial prefactor.

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Three levels, three statuses

Proposition 4 (What is proved: change of frame)

Upward axis, cabin of acceleration a , field giving every body acceleration g :

$$y_{rel} = y - y_c \Rightarrow \ddot{y}_{rel} = g - a, \quad dq_{rel} = m(g - a) dt.$$

Level	Statement	Status
kinematics	$\ddot{y}_{rel} = g - a$	proved (change of frame)
universality	$m_{grav} = m_{inertial}$	empirical (10^{-15})
extension	all of physics, locally	postulated, then tested

The **hypothesis** hides in the words “every body”: the calculation needs g to be universal, and that is Newton’s unexplained coincidence.

Why “principle” is the right word

- The kinematic identity is a theorem — but trivial on its own.
- Its physical content needs $m_{\text{grav}} = m_{\text{inertial}}$, which no argument supplies: it is measured.
- Einstein’s extension to light, clocks and fields is **posed**, not derived. It is judged by its consequences: redshift, light bending, and ultimately general relativity.

Historical arc

Galileo observes the universality → Newton notes the coincidence → Eötvös measures it → Einstein turns it into a principle → Pound–Rebka, GPS and MICROSCOPE confirm its consequences.

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The conjecture as first stated

In the free-falling cabin, $g - a = 0$, so

$$dq_{\text{rel}} = (g - a) dt = 0 \cdot dt.$$

As a Stieltjes measure: $dq_{\text{rel}} = d(0 \cdot t) = 0$. The Jacobian factor linking dq to dt vanishes.

Initial conjecture

“Everything happens as if time did not flow inside the cabin.”

Is this right? The grain tool gives a precise answer.

Analysis: the target axis degenerates, not the source axis

Proposition 5

If $g - a = 0$, the interval $[t, t + dt]$ is sent to $[q_{rel}, q_{rel} + (g - a) dt] = \{q_{rel}\}$: a single point. The Stieltjes measure dq_{rel} is the **zero measure**. The interval dt itself is untouched.

- $dq_{rel} = 0$ means *constant relative momentum*: no force, inertia inside the cabin. It does **not** mean $dt = 0$.
- Relative position still evolves: a ball thrown across the cabin moves in a straight line, which takes time.
- The initial reading confuses the vanishing of a *stratification factor* with the vanishing of the *interval* it stratifies.

The inverse is singular: the momentum clock breaks

Proposition 6

To read time through q we need $q \mapsto t$, with factor $dt/dq = 1/(g - a)$. If $g - a = 0$, this factor is infinite and the inverse is not defined.

Same situation as the warning example

$f(x) = x^2$: $f'(0) = 0$, and $\sqrt{y}$ has a vertical tangent. A zero factor going $t \rightarrow q$ is an infinite factor coming back $q \rightarrow t$.

Time does not stop; **momentum stops being a clock.**

Conjecture 1

The vanishing of a Jacobian factor between time and a quantity X does not signal that time stops. It signals a Cartesian anomaly of the clock X : the inverse $X \mapsto t$ becomes singular and X can no longer parametrise time. A genuine deformation of the time grain is a finite, non-zero factor between two times, $d\tau/dt$; it cannot be read on dq .

Where the intuition finds real content: the horizon

Around a spherical mass M (Schwarzschild), a clock at rest at radius r and a distant observer's time t :

$$d\tau = \sqrt{1 - \frac{r_s}{r}} dt, \quad r_s = \frac{2GM}{c^2}.$$

- Weak field: $\sqrt{1 - r_s/r} \approx 1 + \Phi/c^2$ with $\Phi = -GM/r$ — the redshift of the previous section.
- At the horizon ($r \rightarrow r_s$) the factor tends to 0: seen from far away, the clock *seems* to stop.
- But this is a **coordinate singularity**: a free-falling observer crosses the horizon in finite proper time and notices nothing local.

The conjecture survives

Even where time seems to stop, it is the coordinate t of the distant observer that degenerates as a clock — not proper time. Again a Cartesian anomaly.

Status of the conjecture

Claim	Status
$dq = 0$ is the zero measure; dt untouched	proved
inverse factor $1/(g - a)$ infinite	proved
redshift, Schwarzschild factor	established physics
every vanishing time factor is a clock anomaly	conjecture

How to test it: look at other horizons — the Rindler horizon of the uniformly accelerated frame, cosmological horizons — and check each time that a finite proper time exists where the factor vanishes.

Would the conjecture separate the two masses?

Relax the equality. For each body let $\eta = m_{\text{grav}}/m_{\text{inertial}}$. A body falls with ηg ; the cabin falls with $a = \eta_c g$. For a free body inside:

$$dq_{\text{rel}} = m_{\text{inertial}} (\eta - \eta_c) g dt.$$

$\eta = 1$ for all bodies

The factor vanishes for **every body at once**.
No relative momentum changes: q is a clock for nothing in the cabin. This is the conjecture's situation.

η depends on material

The factor no longer vanishes. Two materials drift apart; q_{rel} grows with time: momentum **becomes a clock again**, slow and material-dependent.

Answer: **coupling, not distinction**. The conjecture presupposes $m_{\text{grav}} = m_{\text{inertial}}$.

What this means for the conjecture

- A momentum clock that started ticking in free fall would be the **signature of a violation**. MICROSCOPE looks exactly for this: two test masses in common free fall, relative drift. None found down to $|\eta_1 - \eta_2| \lesssim 10^{-15}$.
- **General part** (vanishing factor = clock anomaly, not time stopping): a statement about maps and inverses, *independent* of the masses.
- **Physical part** (the factor vanishes for all bodies in free fall): **equivalent** to the weak equivalence principle. It restates it; it does not explain it. Deriving mass equality from it would be circular.
- **Strong version**: the equality extends to gravitational binding energy. A violation would give the Nordtvedt effect (Earth and Moon falling differently toward the Sun). Lunar laser ranging sees none: the Earth–Moon momentum clock does not tick either.

- ① A uniform field gives momentum a **homogeneous** grain against time and a **heterogeneous** grain ($dq/dy = mC/q$) against space.
- ② The homogeneous grain of weight 1 produces Galileo's odd numbers at weight 2.
- ③ Uniform field $\equiv$ accelerated frame; free fall erases it: it is a **connection** Γ , a Cartesian anomaly of the frame.
- ④ Real gravity is the part no frame erases: the **curvature** R (tides).
- ⑤ It all rests on $m_{\text{grav}} = m_{\text{inertial}}$: measured, not proved — hence a *principle*.
- ⑥ $g - a = 0$ does not stop time: it breaks the momentum clock.

A uniform field is an anomaly of the frame;
gravity is the anomaly that no frame removes.

Questions and objections welcome.