Complex theory

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October 2025

Complex numbers from Hamilton

Map from Hamilton :

$$\begin{cases} z.z' &= (x.x' - y.y', x.y' + x'.y) = z'.z, \\ z + z' &= (x + x', y + y') = z' + z. \end{cases}$$

Theorem

In the complex field, $\mathbb{C}$ every number can be written under the form :

$$z = \rho.e^{\imath(\theta + 2k\pi)} = \rho\cos(\theta) + \imath\rho\sin(\theta) = x + \imath y.$$

Argand-Gauss map

$\forall z \in \mathbb{C}$

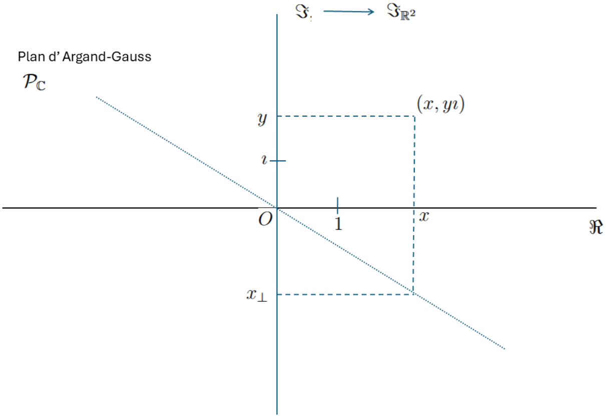

Figure – Deconstruction

$\forall z, z = x + \imath y$

A base on $NE(\mathcal{P}_{\mathbb{C}})$ with one and only one vector

$$z = x + \imath y$$

$$\begin{aligned} z &= -x.(\imath)^2 + y.\imath \\ &= \big[- x\imath + y \big] \imath \\ &= \big[x_{\perp} + y \big] \imath. \end{aligned}$$

NE is the side North-East, which is a space of dimension 2. But we have a vectorial space of dimension 1.

$e^{\imath\theta}$ from the real exponential

Let the functional equation $z^2(\phi) = z(2\phi)$. The solution is $e^{\alpha\phi}$. So in the field $\mathbb{C}$, we consider the solution on $\mathcal{U}_1$, we have $e^{\alpha\phi 2} = \cos(2\phi) + \imath\sin(2\phi)$. With the derivation, and the case $\phi = 0$, we find $\alpha = \imath$.

Let $\theta = 2\phi$, then $\imath e^{i\theta} = \imath^2 \sin\theta + \imath\cos\theta$.

$$\boxed{e^{i\theta} = \cos\theta + \imath\sin\theta}.$$

Let's defining an operator R

Definition (Operator)

An operator is acting on a vectorial space to define another vectorial space.

$R\mathrm{V} = \lambda\mathrm{V}$ with $R = -\imath$ and $\mathrm{V} = (1,0)_{\Delta_y}$. $\lambda = -\imath$. Thus, $-\imath\mathrm{V} = (0,1)$. Where $(0,1) = \imath$. We observe that λ is not real.

The vectorial space $\Im = \Delta_y^{\perp}$ is a distorsion of the vectorial space Δ_y by the action of the operator R.

Construction of $\imath$

Let r an operator $Rot(O, \frac{\pi}{2})$. We apply the transformation $r^2 = r \circ r$ on Δ_x. Then,
$r \circ r(1) = r^2(1,0) = -(1,0) \Leftrightarrow r^2(1,0) = -(1,0) \Leftrightarrow r^2 = -1.$
Hence r^2 is a number, a fortiori r is a number.
We have $r^2 = (-1,0)_{\Delta x} = (1,0)_{\Delta y}$. r is a number such as $r^2 = -1$, and a rotation $Rot(O, \frac{\pi}{2})$. So
$r^{-1} \circ r^2 = r \Leftrightarrow r = r^{-1}(-1,0)_{\Delta x} = r^{-1}(1,0)_{\Delta y} = (0,1)_{\Delta y^{\perp}}.$
We deduce $r = (0,1)$.

Theorem

$\imath$ is a rotate operator $i(O, \frac{\pi}{2})$.

Deconstruction of $\mathcal{P}_{\mathbb{C}}$

Let $\mathcal{P}_{\mathbb{C}} | z = x + \imath y$. We apply a rotation $Rot(O, \frac{\pi}{2})$ to each y of $\Im$.

$$\begin{array}{rcl} \imath : \Im & \to & \Delta y \\ y & \mapsto & \imath y = Rot(O, \frac{\pi}{2})[y] \end{array}$$

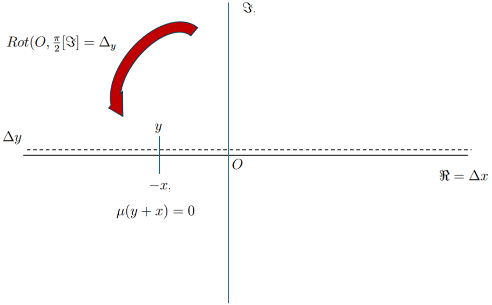

Figure – Deconstruction

Construction of $\mathcal{P}_{\mathbb{C}}$

And deconstruction of 0

We are starting from $\Delta_{x+y} = \Delta_x \cup \Delta_y$ such as $\mu^{\star}_{\Delta_x} + \mu^{\star}_{\Delta_y} = -x + y = 0$.

We notice that $y \longmapsto -y$, We are applying a rotate $r^2 = rot(o, \frac{\pi}{2}) \circ rot(o, \frac{\pi}{2})[y] = -y$, hence $r^2(y) = -y \Leftrightarrow r^2 = -1$. So $\exists r$ as a rotate and a number.

Furthermore $y = -r^2 y = -r(ry) = rot(O, \frac{\pi}{2})[ry] = \Im$.

Thus, $0 \to \mathcal{P}_{\mathbb{C}}$.

Construction of $\mathcal{P}_{\mathbb{C}}$

And deconstruction of 0 - Another approach

So $r = r^{-1}(-1)$. Let's take r as a number on the left side of the equation, and rotate on the right side of the equation. We have $r(1,0)_{\Delta y} = r$, and $-Rot(O, -\frac{\pi}{2})[(1,0)]_{\Delta y} = (0,-1)_{\Delta y^{\perp}}$. Thus $0 \to \mathcal{P}_{\mathbb{C}}^{*}$. As the conjugate map : $\mathcal{P}_{\mathbb{C}}^{*} = 0$ then $\mathcal{P}_{\mathbb{C}}^{*^{*}} = 0^{*} = 0$.

Main theorem

Theorem

$\mathcal{P}_\mathbb{C} = 0.$

From $\mathcal{P}_{\mathbb{R}}^2 to \mathcal{P}_{\mathbb{C}}$

In $\mathcal{P}_{\mathbb{C}}$, $e^{\imath\theta} \circ e^{\imath\phi} = e^{\imath(\theta+\phi)} = e^{\imath\theta} \times e^{\imath\phi}$. So we find an equivalent result in C_1 :
$Rot(O,\theta) \circ Rot(O,\phi) = Rot(O,\theta) \times Rot(O,\phi)$.

Then $Rot(O,\phi)[\mathrm{e}_1] = \cos\ \phi\ \mathrm{e}_1 + \sin\ \phi\ \mathrm{e}_2 \Rightarrow Rot(O,2\phi)[\mathrm{e}_1] = Rot(O,\phi) \circ Rot(O,\phi)[\mathrm{e}_1] = (\cos\ \phi\ \mathrm{e}_1 + \sin\ \phi\ \mathrm{e}_2)^2$.

Pour $\phi = \frac{\pi}{2}, Rot(O, \frac{\pi}{2} + \frac{\pi}{2}) = \mathrm{e}_2^2 = Rot(o,\pi) = -1$.

$$\boxed{\mathrm{e}_2^2 = -1.}$$

Theorem

The real map $\mathcal{P}_{\mathbb{R}^2}$ links to the complex map $\mathcal{P}_{\mathbb{C}}$.

Kind of spaces

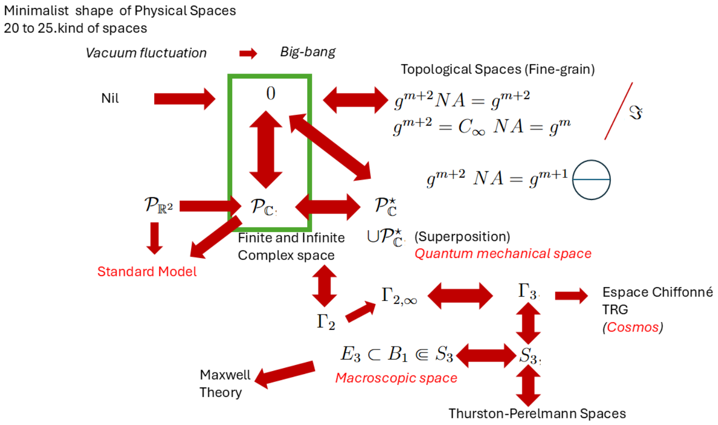

Figure – $\mathcal{F}$. Transformation