Constructing the Integers from the Continuum

Minimal indexing, predecessor and dichotomy ranks in grain theory

Hugues Genvrin

November 2026

Outline

The programme

The framework

The minimal set

The predecessor of ν

The descent cannot be traversed

The integers as dichotomy ranks

Finer observation levels

Status of the construction

Conclusion

Outline

The programme

The framework

The minimal set

The predecessor of ν

The descent cannot be traversed

The integers as dichotomy ranks

Finer observation levels

Status of the construction

Conclusion

Two ways to obtain the integers

Dedekind (1888): going up

- Start from 1 and the successor.
- $\mathbb{N}$ is the smallest chain closed under successor.
- Then $\mathbb{Q}$, then $\mathbb{R}$ by cuts.
- The continuum comes **last**.
- Infinite: in bijection with a strict subset.

Grain theory: going down

- Start from the continuum, given in act.
- The integers: the smallest set that indexes it.
- Described starting from ν: $\nu,\ \nu - 1,\ \nu - 2, \ldots$
- The continuum comes **first**.
- Infinite: indexing the continuum by its subsets.

Same end result, reversed order of priority.

The definition of infinity

Basic definition (paper, introduction)

A set is **infinite** if the set of its subsets can index all the points of a continuum.

The definition of infinity

Basic definition (paper, introduction)

A set is **infinite** if the set of its subsets can index all the points of a continuum.

- The continuum itself satisfies it: its singletons index it.

The definition of infinity

Basic definition (paper, introduction)

A set is **infinite** if the set of its subsets can index all the points of a continuum.

- The continuum itself satisfies it: its singletons index it.
- So there is *at least* one set that indexes the continuum.

The definition of infinity

Basic definition (paper, introduction)

A set is **infinite** if the set of its subsets can index all the points of a continuum.

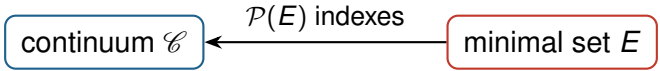
- The continuum itself satisfies it: its singletons index it.
- So there is *at least* one set that indexes the continuum.
- Natural question: what is the **minimal** set that indexes it?

The definition of infinity

Basic definition (paper, introduction)

A set is **infinite** if the set of its subsets can index all the points of a continuum.

- The continuum itself satisfies it: its singletons index it.
- So there is *at least* one set that indexes the continuum.
- Natural question: what is the **minimal** set that indexes it?
- **Starting proposal**: its elements are the *integers*.



Three questions

1. **What does “minimal” mean?**
   The answer depends on the grain of the reference continuum. We fix it, then show that minimality characterises ν.

Three questions

1. **What does “minimal” mean?**
   The answer depends on the grain of the reference continuum. We fix it, then show that minimality characterises ν.

2. **Does the descent reach 1?**
   Yes, but not step by step: along the way there is neither a last unbounded integer nor a first finite one.

Three questions

1. **What does “minimal” mean?**
   The answer depends on the grain of the reference continuum. We fix it, then show that minimality characterises ν.

2. **Does the descent reach 1?**
   Yes, but not step by step: along the way there is neither a last unbounded integer nor a first finite one.

3. **Where does the order of the integers come from?**
   From the order of the dichotomies of the continuum: each integer is a rank of subdivision.

Three questions

1. **What does “minimal” mean?**
   The answer depends on the grain of the reference continuum. We fix it, then show that minimality characterises ν.

2. **Does the descent reach 1?**
   Yes, but not step by step: along the way there is neither a last unbounded integer nor a first finite one.

3. **Where does the order of the integers come from?**
   From the order of the dichotomies of the continuum: each integer is a rank of subdivision.

Then: the observation levels $k\nu$, and the status of the construction.

Outline

The programme

The framework

The minimal set

The predecessor of ν

The descent cannot be traversed

The integers as dichotomy ranks

Finer observation levels

Status of the construction

Conclusion

Skolem's model

- $\mathcal{M}$: a **non-standard** model of Peano arithmetic (Skolem, 1934).
- **Transfer**: $\mathcal{M}$ satisfies every arithmetical statement true in $\mathbb{N}$.
- $\mathbb{N} \subset \mathcal{M}$ denotes the **standard part**.
- Fix a non-standard integer H and set

$$\boxed{\nu = H!}$$

which is divisible by every standard integer.

Not Robinson

We do not go through the non-standard reals: everything is stated with integers and rationals of $\mathcal{M}$.

Internal sets and numerosity

Definition 2.1

- **Internal set**: $\{i \in \mathcal{M} : \text{the } i\text{-th binary digit of } c \text{ is } 1\}$, for some $c \in \mathcal{M}$ (Ackermann coding).
- **Numerosity** $\#E$: number of elements, computed in $\mathcal{M}$.
- For a definable $F \subset \mathbb{N}$: $\#F = \#\big({}^*F \cap [1, \nu]\big)$.

By transfer, finite counting still holds:

- if $\#E = n$, then $\#\mathcal{P}(E) = 2^n$;
- an internal surjection $A \to B$ exists $\iff$ $\#A \geq \#B$ $(B \neq \emptyset)$.

Internal sets are the sets that are "finite as seen from inside $\mathcal{M}$".

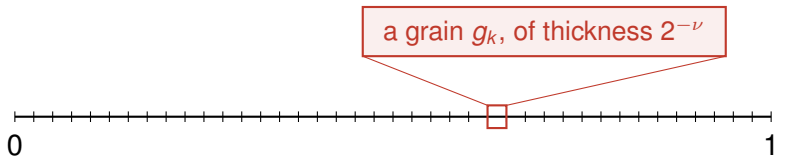
The hypothesis of an actual continuum

Hypothesis (C)

There exists a one-dimensional continuum, given in act, whose grain of individuation has thickness $2^{-\nu}$.

Definition 2.2 (base continuum)

$\mathscr{C}_\nu$: the 2^ν grains $g_k = \left[k\,2^{-\nu}, (k+1)\,2^{-\nu}\right]$, $0 \leq k < 2^\nu$.
More generally, $\mathscr{C}_m$: 2^m grains of thickness 2^{-m}.



Finite, unbounded, infinite

Definition 2.3 (corrected form: infinite is a case of unbounded)

An internal set E is

- **finite** if $\#E$ is standard;
- **unbounded** if it is not finite;
- **infinite** if it is unbounded and there is an internal surjection $\mathcal{P}(E) \to \mathscr{C}_\nu$, that is, if $2^{\#E} \geq 2^\nu$.



For a subset of the window $[1, \nu]$: only the whole window is infinite.

Partial part, proper part

- **Partial part**: F included in the window $[1, \nu]$, at the grain of the integers.
  Evens: $\nu/2$ elements, grain 2 $\Rightarrow$ unbounded.
- **Proper part**: F renumbered as a whole.
  Evens: ν elements, window $[1, 2\nu]$ $\Rightarrow$ infinite.

Not the usual meaning of "proper subset" (strict subset). French: *partie partielle*, *partie propre*.

Remark 2.4

As a partial part, F is infinite $\iff {}^*F \cap [1, \nu] = [1, \nu]$.

For the construction

Minimality is defined from the **partial part** viewpoint: we count in a fixed window, at a fixed grain.

Outline

The programme

The framework

The minimal set

The predecessor of ν

The descent cannot be traversed

The integers as dichotomy ranks

Finer observation levels

Status of the construction

Conclusion

Indexing a continuum

Definition 3.1

An internal set E **indexes** $\mathscr{C}_m$ if there is an internal surjection $\mathcal{P}(E) \to \mathscr{C}_m$.

Proposition 3.2 (minimality)

E indexes $\mathscr{C}_m \iff \#E \geq m$.

The smallest numerosity that indexes the base continuum is ν, attained by

$$\mathbb{N}_\nu = [1, \nu] = \{n \in \mathcal{M} : 1 \leq n \leq \nu\}.$$

Indexing a continuum

Definition 3.1

An internal set E **indexes** $\mathscr{C}_m$ if there is an internal surjection $\mathcal{P}(E) \to \mathscr{C}_m$.

Proposition 3.2 (minimality)

E indexes $\mathscr{C}_m \iff \#E \geq m$.

The smallest numerosity that indexes the base continuum is ν, attained by

$$\mathbb{N}_\nu = [1, \nu] = \{n \in \mathcal{M} : 1 \leq n \leq \nu\}.$$

Proof

$\#\mathcal{P}(E) = 2^{\#E}$ and $\#\mathscr{C}_m = 2^m$: surjection $\iff 2^{\#E} \geq 2^m$. The map $x \mapsto 2^x$ is strictly increasing in $\mathbb{N}$, hence in $\mathcal{M}$ by transfer. Finally $\#\mathbb{N}_\nu = \nu$.

The integers at observation level ν

The **integers at observation level** ν are the elements of $\mathbb{N}_\nu = [1, \nu]$.

- This is the set written $\mathbb{N} = \{1, 2, \ldots, \nu\}$ in the first draft.
- The notation $\mathbb{N}_\nu$ avoids confusion with the standard part $\mathbb{N}$.

Remark 3.3 (the Cantorian counterpart)

For Cantor, the smallest cardinal κ with $2^\kappa \geq \mathfrak{c}$ is $\aleph_0$.

Defining the integers by minimal indexing thus gives **the right size** in both frameworks.

But for Cantor it only yields a *cardinal*, with no order and no successor.

The order will come from the descent, then from the dichotomies.

Outline

The programme

The framework

The minimal set

The predecessor of ν

The descent cannot be traversed

The integers as dichotomy ranks

Finer observation levels

Status of the construction

Conclusion

Characterising $\nu - 1$ by the threshold of infinity

Proposition 4.1

For $n \in \mathcal{M}$, the following are equivalent:

(a) an internal set of numerosity n is unbounded but not infinite, and becomes infinite as soon as one element is added to it;

(b) $n = \nu - 1$.

Characterising $\nu - 1$ by the threshold of infinity

Proposition 4.1

For $n \in \mathcal{M}$, the following are equivalent:

(a) an internal set of numerosity n is unbounded but not infinite, and becomes infinite as soon as one element is added to it;

(b) $n = \nu - 1$.

Proof

(a)⇒(b). E not infinite gives $n < \nu$; $E \cup \{x\}$ infinite gives $n + 1 \geq \nu$. Hence $n = \nu - 1$.

Characterising $\nu - 1$ by the threshold of infinity

Proposition 4.1

For $n \in \mathcal{M}$, the following are equivalent:

(a) an internal set of numerosity n is unbounded but not infinite, and becomes infinite as soon as one element is added to it;

(b) $n = \nu - 1$.

Proof

(a)⇒(b). E not infinite gives $n < \nu$; $E \cup \{x\}$ infinite gives $n + 1 \geq \nu$. Hence $n = \nu - 1$.

(b)⇒(a). If $\nu - 1$ were standard, so would ν be: $\nu - 1$ is unbounded. It is not infinite since $\nu - 1 < \nu$. After adding one element, the numerosity is ν: infinite.

Reading

- We recover, **read backwards**, the remark from *Proper part and partial part*:

$$\mathbb{N} \setminus \{1\}, \text{ of numerosity } \nu - 1, \text{ is unbounded but not infinite.}$$

Reading

- We recover, **read backwards**, the remark from *Proper part and partial part*:

$$\mathbb{N} \setminus \{1\}, \text{ of numerosity } \nu - 1, \text{ is unbounded but not infinite.}$$

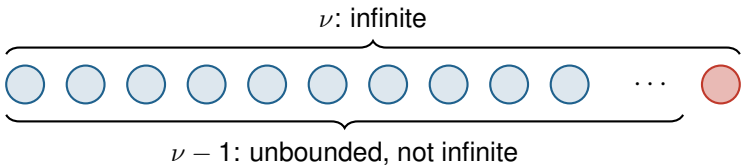
- The predecessor of ν is not set by an arithmetical operation: it is characterised by **the threshold of infinity**.

Reading

- We recover, **read backwards**, the remark from *Proper part and partial part*:

$$\mathbb{N} \setminus \{1\}, \text{ of numerosity } \nu - 1, \text{ is unbounded but not infinite.}$$

- The predecessor of ν is not set by an arithmetical operation: it is characterised by **the threshold of infinity**.
- This is the counterpart of Euclid's principle: the whole is greater than the part, even when only one element is missing.



The descent and the infinity defect

Define $\nu - k$ as the predecessor of $\nu - (k-1)$.

For $k \geq 2$, the one-element characterisation no longer applies as such: it must be generalised.

Definition 4.2

For an internal $E \subset \mathbb{N}_\nu$, the **infinity defect** is $d(E) = \nu - \#E$.

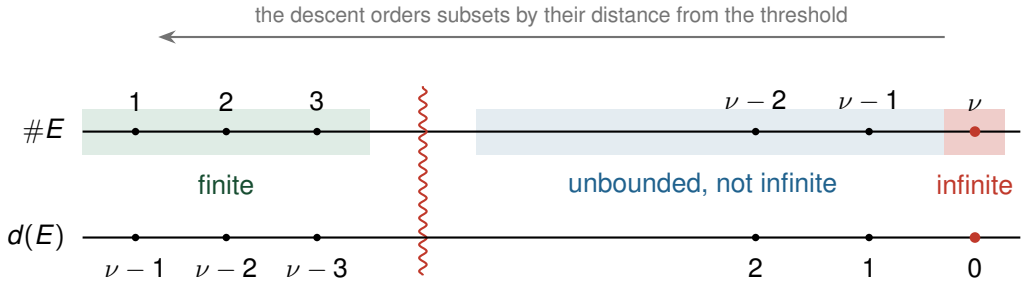
Proposition 4.3

$d(E)$ is the smallest number of elements of $\mathbb{N}_\nu \setminus E$ that must be added to E to make it infinite. Moreover:

- E infinite $\iff$ $d(E) = 0$, i.e. $E = \mathbb{N}_\nu$;
- E finite $\iff$ $\nu - d(E)$ standard;
- E unbounded, not infinite $\iff$ $d(E) \geq 1$ and $\nu - d(E)$ non-standard.

Proof. Adding j elements gives $\#E + j$, infinite $\iff$ $\#E + j \geq \nu$ $\iff$ $j \geq d(E)$.

The defect measures the distance to infinity



- $\nu - k$ is the numerosity of the subsets with **defect** k.
- The descent ends at $\nu - (\nu - 1) = 1 = \#\{\emptyset\}$, a singleton of defect $\nu - 1$.
- To include $0 = \#\emptyset$: the empty subset, of maximal defect ν.

Outline

The programme

The framework

The minimal set

The predecessor of ν

The descent cannot be traversed

The integers as dichotomy ranks

Finer observation levels

Status of the construction

Conclusion

Where does unbounded turn into finite?

The descent starts at ν (unbounded) and ends at 1 (finite). Let us look for the step where it crosses over.

Proposition 5.1 (no crossing in one step)

There is no $k \in \mathcal{M}$ such that $\nu - k$ is unbounded and $\nu - (k+1)$ is finite.

Proof. If $\nu - (k+1) = m$ is standard, then so is $\nu - k = m + 1$.

Where does unbounded turn into finite?

The descent starts at ν (unbounded) and ends at 1 (finite). Let us look for the step where it crosses over.

Proposition 5.1 (no crossing in one step)

There is no $k \in \mathcal{M}$ such that $\nu - k$ is unbounded and $\nu - (k+1)$ is finite.

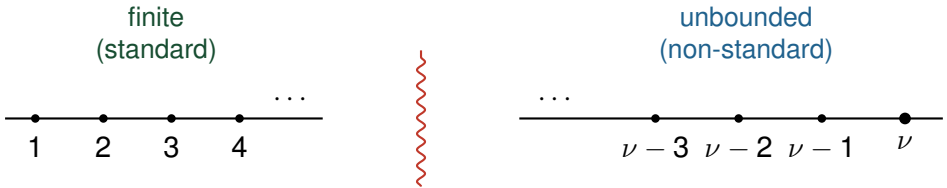
Proof. If $\nu - (k+1) = m$ is standard, then so is $\nu - k = m + 1$.

Corollary 5.2

Every descent of standard length stays among the unbounded: for standard k, $\nu - k$ is unbounded.

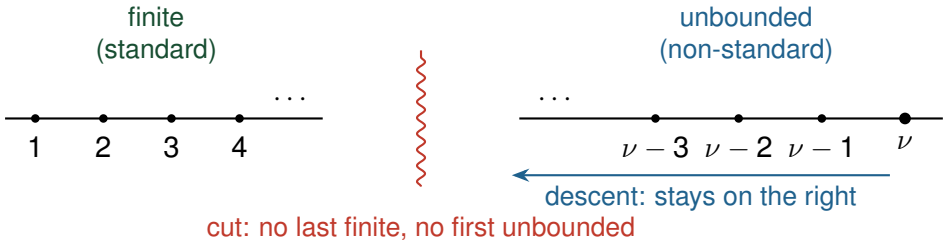
Proof. Otherwise $\nu = (\nu - k) + k$ would be standard.

A cut, not a point



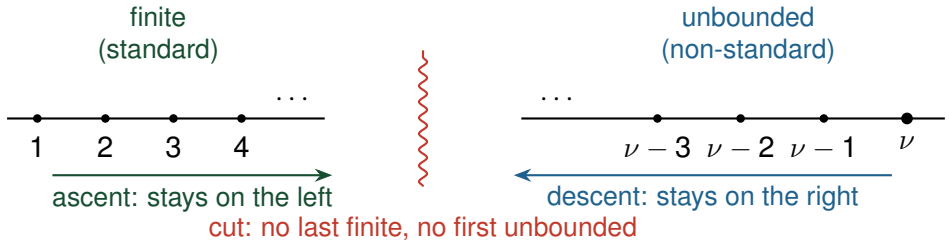
cut: no last finite, no first unbounded

A cut, not a point



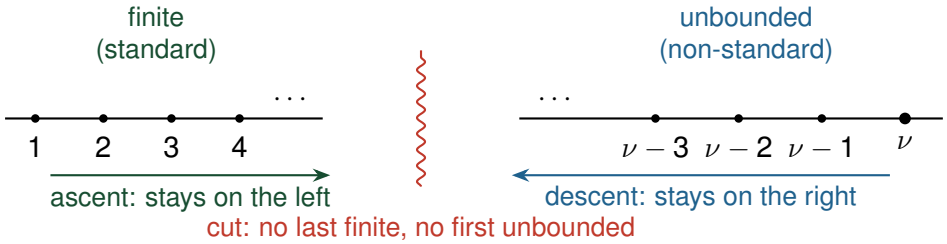
- Each standard-length walk down from ν stays on the unbounded side.

A cut, not a point



- Each standard-length walk down from ν stays on the unbounded side.
- **Remark 5.4.** Symmetrically, going up from 1 in standard steps, one never leaves the finite and never reaches ν.

A cut, not a point



- Each standard-length walk down from ν stays on the unbounded side.
- **Remark 5.4.** Symmetrically, going up from 1 in standard steps, one never leaves the finite and never reaches ν.
- Neither the descent nor the ascent traverses the window step by step.

The cut is invisible from inside

Lemma 5.3 (overspill)

No subset of $\mathcal{M}$ definable in $\mathcal{M}$ (with parameters) is equal to $\mathbb{N}$.

Proof

If a definable D satisfied $D = \mathbb{N}$, we would have $1 \in D$ and $n \in D \Rightarrow n + 1 \in D$. The induction schema, valid in $\mathcal{M}$, would give $D = \mathcal{M}$: contradicting the existence of non-standard elements.

The cut is invisible from inside

Lemma 5.3 (overspill)

No subset of $\mathcal{M}$ definable in $\mathcal{M}$ (with parameters) is equal to $\mathbb{N}$.

Proof

If a definable D satisfied $D = \mathbb{N}$, we would have $1 \in D$ and $n \in D \Rightarrow n + 1 \in D$. The induction schema, valid in $\mathcal{M}$, would give $D = \mathcal{M}$: contradicting the existence of non-standard elements.

Consequence

The predicate “finite” is **external**: no formula of arithmetic expresses it.

Seen from $\mathcal{M}$, the descent $\nu, \nu - 1, \ldots, 2, 1$ is an internal sequence of ν terms, with no visible moment where unbounded becomes finite.

Consequence: the window is actual

- The induction from ν down to 1 has $\nu - 1$ steps: a **non-standard** number.
- It is legitimate as an **internal induction**: the sequence $(\nu - k)_{0 \leq k \leq \nu - 1}$ exists in $\mathcal{M}$ by induction and transfer.
- It cannot be carried out from outside, step after step.

Consequence: the window is actual

- The induction from ν down to 1 has $\nu - 1$ steps: a **non-standard** number.
- It is legitimate as an **internal induction**: the sequence $(\nu - k)_{0 \leq k \leq \nu - 1}$ exists in $\mathcal{M}$ by induction and transfer.
- It cannot be carried out from outside, step after step.

Interpretation

The descent **does not generate** $\mathbb{N}_\nu$: it **describes** it.

The window $[1, \nu]$ is given all at once, together with the continuum it indexes.

Consequence: the window is actual

- The induction from ν down to 1 has $\nu - 1$ steps: a **non-standard** number.
- It is legitimate as an **internal induction**: the sequence $(\nu - k)_{0 \leq k \leq \nu - 1}$ exists in $\mathcal{M}$ by induction and transfer.
- It cannot be carried out from outside, step after step.

Interpretation

The descent **does not generate** $\mathbb{N}_\nu$: it **describes** it.

The window $[1, \nu]$ is given all at once, together with the continuum it indexes.

This is not a weakness: a potential infinity would be one that can be traversed. This one cannot be traversed: that is exactly the thesis of an **actual infinity**.

Outline

The programme

The framework

The minimal set

The predecessor of ν

The descent cannot be traversed

The integers as dichotomy ranks

Finer observation levels

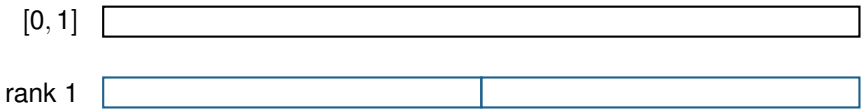
Status of the construction

Conclusion

Reading the order in the continuum

The descent provides an order, but by means of the predecessor, an *arithmetical operation*.

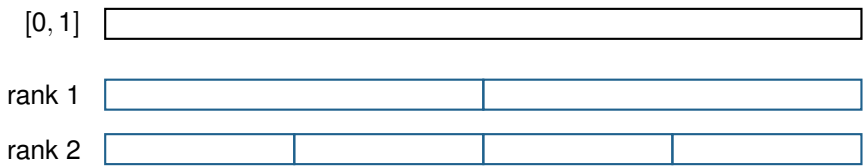
This order can be read directly in the continuum.



Reading the order in the continuum

The descent provides an order, but by means of the predecessor, an *arithmetical operation*.

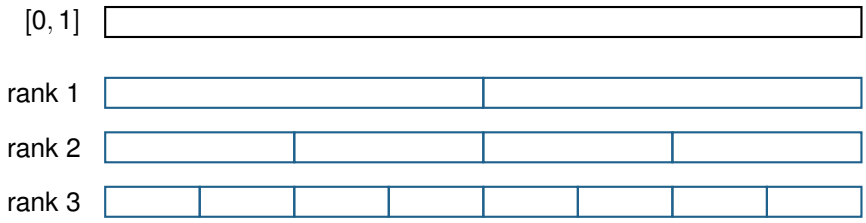
This order can be read directly in the continuum.



Reading the order in the continuum

The descent provides an order, but by means of the predecessor, an *arithmetical operation*.

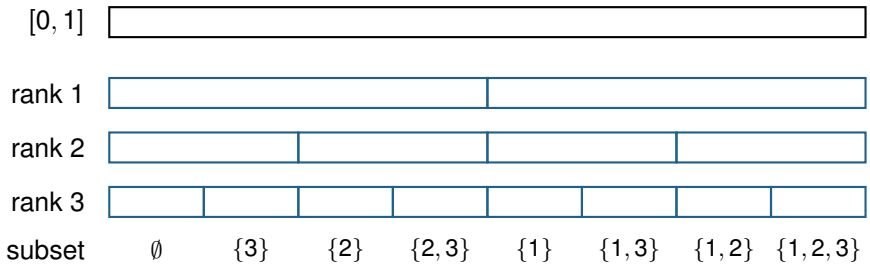
This order can be read directly in the continuum.



Reading the order in the continuum

The descent provides an order, but by means of the predecessor, an *arithmetical operation*.

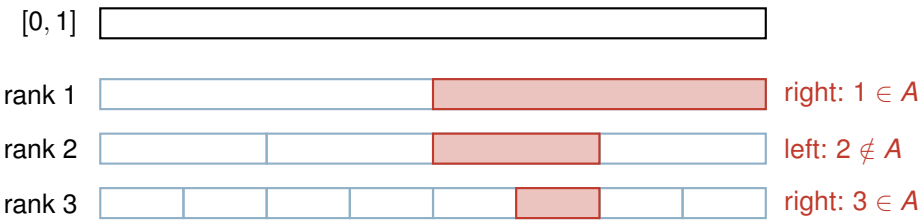
This order can be read directly in the continuum.



At rank i: cells of thickness 2^{-i}. An **address** = the set of ranks where the right half was taken.

(Three ranks instead of ν, as an illustration.)

Example: the grain with address $\{1, 3\}$



subset A	digits $a_1 a_2 a_3$	$k_A = \sum_{i \in A} 2^{3-i}$	grain
$\{1, 3\}$	101	$4 + 1 = 5$	$[5/8,\ 6/8]$

Subsets and grains: a bijection

Proposition 6.1

The map

$$\beta : \mathcal{P}(\mathbb{N}_\nu) \longrightarrow \mathscr{C}_\nu, \qquad A \longmapsto g_{k_A}, \qquad k_A = \sum_{i \in A} 2^{\nu - i},$$

is an internal bijection. The grain $\beta(A)$ has left endpoint $\sum_{i \in A} 2^{-i}$.

Proof

Every k with $0 \leq k < 2^\nu$ is written uniquely as $k = \sum_{i=1}^{\nu} a_i \, 2^{\nu - i}$, $a_i \in \{0, 1\}$: its base-2 expansion on ν digits, valid in $\mathcal{M}$ by transfer. It is assigned $A = \{i : a_i = 1\}$.

Subsets and grains: a bijection

Proposition 6.1

The map

$$\beta : \mathcal{P}(\mathbb{N}_\nu) \longrightarrow \mathscr{C}_\nu, \qquad A \longmapsto g_{k_A}, \qquad k_A = \sum_{i \in A} 2^{\nu - i},$$

is an internal bijection. The grain $\beta(A)$ has left endpoint $\sum_{i \in A} 2^{-i}$.

Proof

Every k with $0 \leq k < 2^\nu$ is written uniquely as $k = \sum_{i=1}^{\nu} a_i \, 2^{\nu - i}$, $a_i \in \{0, 1\}$: its base-2 expansion on ν digits, valid in $\mathcal{M}$ by transfer. It is assigned $A = \{i : a_i = 1\}$.

Key point

A subset of $\mathbb{N}_\nu$ is the address of a grain.

Up to the order of the digits, β is the Ackermann coding, read geometrically.

Why base 2?

- A subset $A \subset E$ is a sequence of **yes** / **no** choices: each element is in or out.
- Indexing by the **set of subsets** therefore means indexing by binary sequences.
- The dichotomy is not a matter of convenience: it is **imposed by the definition of infinity**.

Why base 2?

- A subset $A \subset E$ is a sequence of **yes** / **no** choices: each element is in or out.
- Indexing by the **set of subsets** therefore means indexing by binary sequences.
- The dichotomy is not a matter of convenience: it is **imposed by the definition of infinity**.

Counter-check: a ternary subdivision

A continuum with 3^m grains requires a set of numerosity n with $2^n \geq 3^m$, i.e. $n \approx m \log_2 3$.

The ternary ranks no longer correspond one-to-one to the elements of the minimal set.

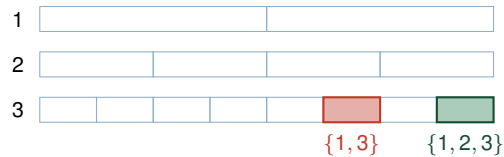
This is why the grain thickness is taken of the form $2^{-\nu}$.

Every rank is necessary

Proposition 6.2 (minimality of the ranks)

For every internal $R \subsetneq \mathbb{N}_\nu$, restricting addresses to the ranks in R is not injective: no strict set of ranks individuates the grains.

Proof. If $j \notin R$, the grains $\beta(A)$ and $\beta(A \triangle \{j\})$ are distinct but have the same digits on R.

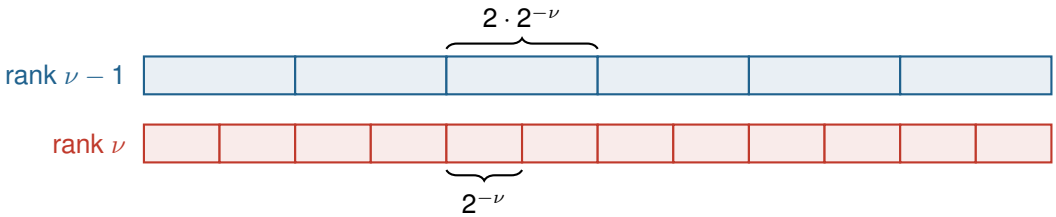


With $R = \{1, 3\}$: the two grains have the same digits at ranks 1 and 3. Only rank 2 separates them.

Geometric version of Proposition 3.2: the integers are the necessary **independent dichotomies**, with none to spare.

The geometric meaning of $\nu - 1$

- Ranks $1, \ldots, \nu - 1$ individuate the cells of thickness $2^{-(\nu-1)} = 2 \cdot 2^{-\nu}$.
- These are the grains of $\mathscr{C}_{\nu-1}$: each contains **two** grains of $\mathscr{C}_\nu$.
- So $[1, \nu - 1]$ indexes the continuum at a **double grain**, but not the base continuum.



Translation of Proposition 4.1: a set of numerosity $\nu - 1$ is unbounded but not infinite because it lacks **exactly one dichotomy**.

The order of the integers is the order of refinement

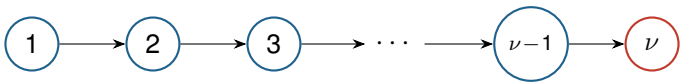
- $i < j \iff$ the dichotomy of rank i is coarser than that of rank j;
- $j = i + 1 \iff$ the cells of rank j are the halves of those of rank i.

What we gain

Order and successor are **read off** the nested structure of the continuum, instead of being postulated.

What we assume

The nested dyadic structure itself, with its last rank ν.



from coarsest to finest

Outline

The programme

The framework

The minimal set

The predecessor of ν

The descent cannot be traversed

The integers as dichotomy ranks

Finer observation levels

Status of the construction

Conclusion

Continuing the dichotomies

Nothing forces us to stop at rank ν. For $1 \leq k \leq \nu$, consider $\mathscr{C}_{k\nu}$, with grain $2^{-k\nu}$.

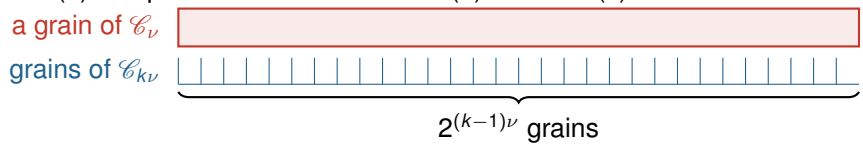
Proposition 7.1

(a) The smallest set that indexes $\mathscr{C}_{k\nu}$ has numerosity $k\nu$: it is $[1, k\nu]$.

(b) If $k \geq 2$, $\mathbb{N}_\nu$, as a partial part of $[1, k\nu]$, is **unbounded but not infinite** relative to $\mathscr{C}_{k\nu}$.

(c) Each grain of $\mathscr{C}_\nu$ is the union of $2^{(k-1)\nu}$ grains of $\mathscr{C}_{k\nu}$.

Proof. (a): Proposition 3.2 with $m = k\nu$. (b): $\nu < k\nu$. (c): $2^{-\nu} = 2^{(k-1)\nu} \cdot 2^{-k\nu}$.



The minimal set depends on the level

level	grain thickness	minimal set	status of $\mathbb{N}_\nu$
ν	$2^{-\nu}$	$[1, \nu]$	**infinite**
2ν	$2^{-2\nu}$	$[1, 2\nu]$	unbounded, not infinite
$k\nu$	$2^{-k\nu}$	$[1, k\nu]$	unbounded, not infinite ($k \geq 2$)
ν^2	$2^{-\nu^2}$	$[1, \nu^2]$	unbounded, not infinite

Exact statement of the construction

The **integers at observation level** ν are the elements of the smallest set that indexes the continuum whose grain has thickness $2^{-\nu}$.

Level ν^2 is that of $\mathbb{N}^2$, used for the rationals: the same set is infinite at the grain of $\mathbb{N}$ and unbounded at the grain of $\mathbb{N}^2$.

The relativity of infinity

No set of integers is infinite in the absolute.
It is infinite **relative to a grain**.

- The choice of ν is not imposed by the continuum alone: it is a **choice of observation level**.
- Refining the grain enlarges the minimal set, and demotes the former set to merely unbounded.
- We recover the structure of *Proper part and partial part*:
  - **infinite** is a threshold, without degrees;
  - **unbounded** is graded, and its degree is the numerosity: $\nu < 2\nu < \nu^2 < \cdots$

Outline

The programme

The framework

The minimal set

The predecessor of ν

The descent cannot be traversed

The integers as dichotomy ranks

Finer observation levels

Status of the construction

Conclusion

What is primitive

The construction does not start from nothing. It assumes:

- **hypothesis (C)**: the continuum and its grain of thickness $2^{-\nu}$;
- the **model** $\mathcal{M}$: the number ν, internal sets, counting by transfer.

What is primitive

The construction does not start from nothing. It assumes:

- **hypothesis (C)**: the continuum and its grain of thickness $2^{-\nu}$;
- the **model** $\mathcal{M}$: the number ν, internal sets, counting by transfer.

A circularity to acknowledge

ν appears as early as the definition of the base continuum, hence of infinity. The integers of the window are not built without the integers of $\mathcal{M}$: they are **reinterpreted**.

What is primitive

The construction does not start from nothing. It assumes:

- **hypothesis (C)**: the continuum and its grain of thickness $2^{-\nu}$;
- the **model** $\mathcal{M}$: the number ν, internal sets, counting by transfer.

A circularity to acknowledge

ν appears as early as the definition of the base continuum, hence of infinity. The integers of the window are not built without the integers of $\mathcal{M}$: they are **reinterpreted**.

What changes

The order of priority: the integers are not laid down to build the continuum; they are obtained as **what the continuum needs in order to be indexed**.

In what sense the construction is non-axiomatic

It uses

- the set of subsets;
- numerosity;
- adding an element;
- internal induction;
- transfer.

All of this belongs to Peano arithmetic and its non-standard model.

It does not postulate

- the successor;
- the order;
- minimality in Dedekind's sense.

It **derives** them from the sole requirement of indexing a given continuum, at a given grain.

Accurate formulation

The construction **adds no axiom specific to the integers**.

Descent and ascent

	Dedekind	**Grain theory**
starting point	1	ν
movement	ascent (successor)	descent (predecessor)
minimality	chain closed under successor	set that indexes the continuum
first step	$2 = 1 + 1$, postulated	$\nu - 1$, given by the threshold
order	postulated	read off the dichotomies
traversal of the window	not step by step	not step by step

The advantage of the descent

Its starting point is characterised by a property (indexing the continuum), and its first step by the threshold of infinity. The ascent starts from 1 without saying anything about it.

The status of the continuum

Hypothesis (C) is a hypothesis

- **Mathematically**, the existence of a continuum is relative to the chosen framework.
- **Physically**, measurements always have finite precision: no experiment distinguishes an actual continuum from a sufficiently fine discrete structure.

The status of the continuum

Hypothesis (C) is a hypothesis

- **Mathematically**, the existence of a continuum is relative to the chosen framework.
- **Physically**, measurements always have finite precision: no experiment distinguishes an actual continuum from a sufficiently fine discrete structure.

Grain theory accommodates both readings

The continuum is always taken at some resolution.

A grain of thickness $2^{-\nu}$ is infinitesimal, and its real shadow is the whole segment $[0, 1]$.

Outline

The programme

The framework

The minimal set

The predecessor of ν

The descent cannot be traversed

The integers as dichotomy ranks

Finer observation levels

Status of the construction

Conclusion

Three results

1. **Minimality and threshold.** ν is the smallest numerosity that indexes the base continuum; $\nu - 1$ is the unique unbounded, non-infinite numerosity that a single element makes infinite. The descent measures an *infinity defect*.

Three results

1. **Minimality and threshold.** ν is the smallest numerosity that indexes the base continuum; $\nu - 1$ is the unique unbounded, non-infinite numerosity that a single element makes infinite. The descent measures an *infinity defect*.
2. **Cut.** The descent never crosses, in one step, the boundary between unbounded and finite, which is external. The window $[1, \nu]$ is *given in act*; the descent describes it.

Three results

1. **Minimality and threshold.** ν is the smallest numerosity that indexes the base continuum; $\nu - 1$ is the unique unbounded, non-infinite numerosity that a single element makes infinite. The descent measures an *infinity defect*.
2. **Cut.** The descent never crosses, in one step, the boundary between unbounded and finite, which is external. The window $[1, \nu]$ is *given in act*; the descent describes it.
3. **Dichotomies.** Each integer is the rank of a dichotomy. A subset of $[1, \nu]$ is the address of a grain, no rank is superfluous, and the order of the integers is the order of refinement.

Three results

1. **Minimality and threshold.** ν is the smallest numerosity that indexes the base continuum; $\nu - 1$ is the unique unbounded, non-infinite numerosity that a single element makes infinite. The descent measures an *infinity defect*.
2. **Cut.** The descent never crosses, in one step, the boundary between unbounded and finite, which is external. The window $[1, \nu]$ is *given in act*; the descent describes it.
3. **Dichotomies.** Each integer is the rank of a dichotomy. A subset of $[1, \nu]$ is the address of a grain, no rank is superfluous, and the order of the integers is the order of refinement.

The infinity of the integers is not absolute: it is relative to the grain of the continuum they index.

Open questions for discussion

- **Dimension 2.** The square $[0,1]^2$ at resolution $-\nu$ has $2^{2\nu}$ grains: the minimal set has numerosity 2ν. How does this relate to the window $[1,\nu]^2$ of the rationals?
- **Zero.** $0 = \#\emptyset$ is the subset of maximal defect. Should it be included among the constructed integers, or kept as a bound?
- **The choice of H.** The construction does not use the divisibility of $\nu = H!$: it holds for any non-standard ν. Divisibility is only needed in the other papers (grain of the evens, of the multiples).
- **The absolute point.** If the absolute point is a grain, is the last rank ν a fact about the continuum, or only about observation?

References

- V. Benci, M. Di Nasso, “Numerosities of labelled sets: a new way of counting”, *Advances in Mathematics*, 173 (2003), 50–67.
- G. Cantor, “Über eine elementare Frage der Mannigfaltigkeitslehre”, *Jahresbericht der DMV*, 1 (1891), 75–78.
- R. Dedekind, *Was sind und was sollen die Zahlen?*, Vieweg, 1888.
- R. Kaye, *Models of Peano Arithmetic*, Oxford Logic Guides 15, Clarendon Press, 1991.
- T. Skolem, “Über die Nicht-charakterisierbarkeit der Zahlenreihe. . . ”, *Fundamenta Mathematicae*, 23 (1934), 150–161.
- H. Genvrin, *Théorie du grain* [Grain theory], manuscript, in French.
- H. Genvrin, “Partie propre et partie partielle chez les entiers” [Proper part and partial part among the integers], manuscript, in French, 2026.
- H. Genvrin, “Numérosité des rationnels” [Numerosity of the rationals], manuscript, in French, 2026.
- H. Genvrin, “La construction des entiers à partir du continu” [Constructing the integers from the continuum], manuscript, in French, 2026.