Gauges

A relational alternative to vector calculus

Hugues Genvrin

2014 – Theory + Applications

Outline

1. Motivation
2. Algebraic formalization
3. Portability of gauging
4. Stratification
5. Application: molecular deformation
6. Application: adaptive loop
7. Conclusion

The guiding idea

Substituting an ancestor for vector structures

Vector spaces presuppose a heavy hierarchy (groups, fields, bases) and freeze time and space as *absolutes*, a source of avoidable differential complexity.

- A gauge starts over from the most primitive act: *to order* and *to identify*.
- Basic building blocks: an *order relation* and an *equivalence relation* — nothing more.
- Epistemological stance: *Archimedes* (pragmatism) against reductionist universality.

Guiding thread

From circadian time (biology) to the photon (physics), one and the same grid: *what orders, and what is equivalent?*

Generic and strict order

Definition 1 (Generic order)

$\mathscr{O} = (\mathscr{O}_1, \mathscr{O}_2)$ is a generic antisymmetric pre-order:

$$(x\,\mathscr{O}_1\,y) \;\wedge\; (y\,\mathscr{O}_1\,x) \;\Leftrightarrow\; x\,\mathscr{R}\,y,$$

where $\mathscr{R}$ (symmetric) records non-orderability.

Definition 2 (Strict order)

The strict order is the complement of the generic order $\mathscr{O}$, denoted $\overline{\mathscr{O}}$.

Note that the complement of the strict order is a generic order.

Gauge

Definition 3 (Gauge)

On $F \subset E$, the pair $\mathscr{J}_F(\mathscr{O}, \mathscr{R})$ such that

$$(x\, \overline{\mathscr{O}}\, y \;\wedge\; y\, \overline{\mathscr{O}}\, x) \;\Leftrightarrow\; x\, \mathscr{R}\, y.$$

Key notation (total pre-order)

$x \preceq y :\Leftrightarrow (x\, \mathscr{O}_1\, y) \vee (x\, \mathscr{R}\, y)$. $\mathscr{R}$ = symmetric part, $\mathscr{O}_1$ = asymmetric part. The gauge property $\equiv$ "$\preceq$ is total".

Three scopes of a gauge

Absolute

$F = E$: the gauge covers its entire native set.

Of capacity

$F \subset E$, $F' \subset E'$ with $E \cap E' = \varnothing$: a gauge of F also gauges F'.

Universal

Absolute *and* of capacity relative to all the E_i.

Comparing gauges: sphere and fineness

- **Compatibility:** a morphism α expresses the elements of one set in the other. A *sphere* $\mathscr{S}$ gathers compatible sets.
- **Fineness:** $\mathscr{J}_i$ is *finer* than $\mathscr{J}_k$ if it orders more —

$$\#o(\widehat{E_k}, \mathscr{J}_i) \;\geq\; \#o(\widehat{E_k}, \mathscr{J}_k),$$

strict for *heterogeneous* gauging, equal for *homogeneous* (isofine if over the whole sphere).

Order $\mathscr{O}_\delta$ on gauges

"finer than" is an order relation (antisymmetric + transitive) on the sphere: gauges form a hierarchy.

Gauge morphism (without bijectivity)

Definition 4 (Morphism)

$\alpha : E \to E'$ *preserves the pre-order:* $x \preceq y \Rightarrow \alpha(x) \preceq' \alpha(y)$.

- Preserves equivalence; may *collapse* a strict into an equivalence.
- Neither injectivity nor surjectivity is required — it is this looseness that gives the theorems content.

The pitfall avoided

"Morphism = bijection" would make invariance *tautological*. We therefore separate transport from invertibility.

Pullback: the robust direction

Theorem 5 (Gauge induced by pullback)

For any map $\alpha : E \to E'$ and any gauge $\mathscr{J}_{E'}$,

$$x \,\mathscr{O}_1\, y :\Leftrightarrow \alpha(x)\,\mathscr{O}_1'\,\alpha(y), \qquad x \,\mathscr{R}\, y :\Leftrightarrow \alpha(x)\,\mathscr{R}'\,\alpha(y)$$

defines a gauge on E, of which α is a morphism.

Scope

No assumption on α. Folding, projecting, crushing: a gauge is always brought back. The only requirement: the target is gauged. No α^{-1} is involved.

Pushforward: the conditional direction

Lemma 6 (Strict/equivalence compatibility)

In a total pre-order: $a \mathscr{R} a'$, $b \mathscr{R} b'$, $a \mathscr{O}_1 b \Rightarrow a' \mathscr{O}_1 b'$.

Theorem 7 (Pushed-forward gauge)

α a morphism. The pushforward on $\alpha(E)$ is a gauge if and only if the fibers saturate the equivalence:

$$\alpha(x) = \alpha(y) \;\Rightarrow\; x \mathscr{R} y \qquad (\ker \alpha \subseteq \mathscr{R}).$$

Reading

A morphism may *merge equivalent classes*, never *crush two ordered elements*. Automatic if α is injective.

Monotonicity of fineness

Theorem 8 (Decrease of fineness)

$\bar{\alpha} : E/\mathscr{R} \to \alpha(E)/\mathscr{R}'$ is a surjective order morphism:

$$\#(\alpha(E)/\mathscr{R}') \;\le\; \#(E/\mathscr{R}),$$

equality iff $\bar{\alpha}$ is injective.

The real invariant

Not "the gauge is conserved", but "it is conserved or degrades". Gauging can only *coarsen* along a map, never refine.

Bijective case and topological signature

Corollary 9 (Strict invariance)

α bijective with morphism inverse (homeomorphism): pullback and pushforward coincide, isofine gauges, invariance in the full sense.

Corollary 10 (Drop in fineness = topological event)

A gluing that identifies two ordered elements violates $\ker \subseteq \mathscr{R}$: fineness drops on the seam, just as the Euler–Poincaré characteristic varies.

Möbius / tube

No longer an exception: it is an instance of the monotonicity of fineness.

Ontology: the vocabulary of transformations

Definitions

Congruence: same class modulo $\mathscr{R}$.
Equality: congruence according to *all* attributes.
Identity: automorphism $x \mapsto x$.
Meism: an isomorphism carrying an entity of one set to a *distinct element* of another.

Role of meism

An ontological bridge between $(E, F, \mathscr{I}_E)$ and $(F, E, \mathscr{I}_F)$ that preserves both static and dynamic, without positing an "original" set.

The quantum and its dimension

Definition 11 (Quantum)

A gauging multiplicity on E — in the sense of Kant (an object endowed with magnitude) and of Riemann (a determinable manifold).

Definition 12 (Dimension of a quantum)

Dimension of the quotient space of the canonical projection of its perfect determination.

Modes of determination (Cantor, Borel)

Ordering, counting, measure: as many ways of building the capacity gauge.

Topological layer: locus and situs

Locus (local)

Position of an element. Convergence, neighborhood, *continuity* as minimal topological dimension; a *continuum* is such at every interior point.

Situs (global)

Homogeneous characters imposed by the quantum: constant curvature (spherical, hyperbolic, Euclidean) vs. heterogeneous (general relativity).

Topological compatibility gauge

Connects different situs — discrete $\leftrightarrow$ continuous — via partitions $\mathscr{I}(\widehat{x}, \widehat{y})$.

Differential layer

Definition 13 (Relational differentiability)

α is differentiable at e_0 if $\dfrac{\mu(\alpha(E_i))}{\mu(E_i)}$ converges on the left ($\mathscr{O}_1$) and on the right ($\mathscr{O}_2$), with the same value.

Differentiation vs. derivative

Working only with the derivative amounts to positing an *absolute gauge of the trend*: the complexity bias.

Analytic layer: $\mathbb{R}$, the universal gauge

Theorem 14 (Analytic gauge by real quantification)

For any E and any map $\alpha : E \to \mathbb{R}$, the pullback of $\mathscr{J}(\leq, =)$ gauges E; α thereby becomes a morphism.

- **Measure** $\mu : \mathscr{E}' \to [0, +\infty]$: a gauge on the σ-algebra (the *subsets*).
- To descend to *points*: density f, cumulative distribution function, or observable $g : F \to \mathbb{R}$.
- **Composition:** $g \circ \alpha : E \to \mathbb{R}$ gauges E with no intermediate structure.

Metric layer and the pyramid

Metric gauge

A pair $(\mathscr{R}, \mathscr{O})$ + an isomorphism onto a quantification set (standard, norm, probability measure) to measure gaps and distances.

set-theoretic $\rightarrow$ topological $\rightarrow$ differential $\rightarrow$ analytic $\rightarrow$ metric

The pyramid of gauges

Assuming the levels implicitly (differentiating in space / time) *often works* but introduces biases over a broad class of cases. The explicit method makes them visible.

Three gauges on the configuration space $\mathscr{C}$

By pullback of three real quantifications

$\mathscr{J}_V$ (energy V), $\mathscr{J}_\xi$ (reaction coordinate ξ), $\mathscr{J}_\sigma$ (geometric parameter σ).

Theorem 15 (Transversality — over all of $\mathscr{C}$)

Level sets $\{V{=}c\}$, $\{\xi{=}c\}$, $\{\sigma{=}c\}$ transverse $\Rightarrow$ pairwise incomparable gauges ($\mathscr{R}_\delta$).

Sisters under a mother gauge

All dominated by the configuration gauge $q \mapsto q$, which is their *superposition* (least upper bound of fineness).

Along the path γ: a chain of fineness

Theorem 16 (Hierarchy along the path)

Over a barrier: $\xi \circ \gamma$ monotone (injective), $V \circ \gamma$ folds ($V(q_-) = V(q_+)$, $\xi(q_-) < \xi(q_+)$). Hence $\mathscr{J}_\xi \, \mathscr{O}_\delta \, \mathscr{J}_V$, strict at the fold.

Corollary 17

$\mathscr{J}_\xi$ is the optimal gauge (maximal capacity + fineness); V is a degraded optimal gauge.

Theorem 18 (Bond breaking)

Connectivity $\mathscr{J}_\kappa$ violates $\ker \subseteq \mathscr{R}$ at the transition state: drop in fineness = signature of the chemical event.

Reading the figure (three panels, ξ axis)

- $\mathscr{I}_\xi$: monotone line — maximal fineness.
- $\mathscr{I}_V$: barrier; q_-, q_+ at the same level — fold.
- $\mathscr{I}_\kappa$: steps; drop at the transition state.

Three preserved properties

ξ: monotonicity.
V: regularity.
κ: connectivity.
One and the same event, three distinct responses.

The sensorimotor loop

Two pathways, not one arrow

$a : I \to \Phi$ (efferent, action), $p : \Phi \to I$ (afferent, perception).
Functional space Φ gauged by value/viability.

Definition 19 (Internal gauge = value landscape)

$\mathscr{J}_I := a^* \mathscr{J}_\Phi : \; x \preceq_I y \Leftrightarrow a(x) \preceq_\Phi a(y).$

What is automatic

The value order, by pullback, with no prior structure on I. But it is a *landscape*, not a *trajectory*.

Three conditions, three consequences

Coherence

p a morphism $\Rightarrow T = p \circ a$ a morphism: the loop does not *reverse* value.

Ascent (Lyapunov)

$x \preceq_I T(x) \Rightarrow$ *ascending* trajectories; a fixed point is an adapted state (maximal).

Stability

$T^* \mathscr{I} = \mathscr{I}$ (T an order embedding): the gauge is a *fixed point of its own induction*. Adaptation = a gauge stable under the loop.

Discrimination bound and theoretical neighborhood

Theorem 20 (Bounded internal fineness)

$\bar{a} : I/\mathscr{R}_I \hookrightarrow \Phi/\mathscr{R}_\Phi$ *injective:*

$$\#(I/\mathscr{R}_I) \ \leq \ \#(\Phi/\mathscr{R}_\Phi).$$

Falsifiable prediction

The organism distinguishes only what its functional reading distinguishes: categorical perception, sensory thresholds.

Situation

Generalizes Ashby (ultrastability) and Aubin (viability). Own contribution: a *hierarchy* of gauges + a *generic order* that preserves incomparable trade-offs (survival vs. reproduction).

What the framework brings

- A *correct invariant*: gaugeability transports by pullback; fineness only decreases.
- A unified *signature*: a drop in fineness = a topological event (Möbius, transition state).
- A *generic order* that embraces incomparability — a descriptive virtue in physics (transverse situs) as in biology (trade-offs).
- An explicit *stratification* that flushes out the complexity bias of implicit differential models.

Open directions

- Strengthen the stability theorem: a compactness / continuity assumption on T to guarantee $K = \bigcap_n T^n(I) \neq \varnothing$.
- Loop: compose the two pullbacks in opposite directions and seek the self-consistent gauge on the attractor.
- Photon: reformulate interference *photon by photon* (a single photon interferes with itself), not as a two-body interaction.
- Complementary figure: transverse foliations of $\mathscr{C}$ illustrating off-path incomparability.