

# Grains in Motion

Free Fall, Galileo's Inclined Plane and the Brachistochrone  
in Grain Theory

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Two studies, one question: *what does grain theory say about real motion?*

## 1. Free fall and Galileo's inclined plane

- what an experiment can and cannot prove about the continuum
- Galileo's odd-number law as an **exact** identity between hyperintegers
- scale invariance, power laws, the historical experiment

## 2. The brachistochrone in layers of grains

- Bernoulli's optical construction, with grains as layers
- Snell's law as an **exact** optimality condition in each grid
- the cycloid as a promotion; Galileo at the infinitesimal scale; Huygens' tautochrone

In both cases: the grid is **sharp**, and **promotion** gives back classical mechanics.

Grain theory: reminders

I. Free fall

II. The brachistochrone in layers of grains

Synthesis

## Grain theory: reminders

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# Grids, grains and promotion

Let  $n_\infty$  be a hyperinteger,  $k \geq 1$  and  $\varepsilon = 2^{-kn_\infty}$  (infinitesimal).

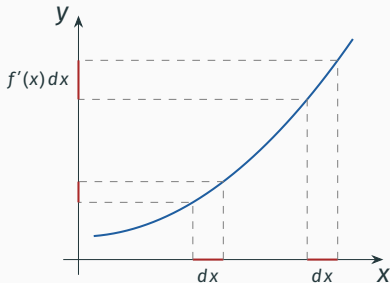
- The grid of resolution  $-kn_\infty$  splits  $[0, 1[$  into  $2^{kn_\infty}$  **grains**  $[m\varepsilon, (m+1)\varepsilon[$ .
- Their endpoints  $m\varepsilon$  are the **junctions**: dyadic hyperrationals.
- **Promotion**  $\tilde{\mathbb{R}}_{\text{fin}}/\mathfrak{m} = \mathbb{R}$  is realized by the standard part **st**: infinitesimals collapse to **0**.
- **Transfer**: first-order statements true for all integers remain true for hyperintegers.



# Functional stratification

## Theorem 1 (Functional stratification)

A differentiable  $f$  maps the uniform interval  $[x, x + dx]$  of its source axis onto  $[f(x), f(x) + f'(x) dx]$  of its target axis: it acts on the target axis as a **local stratification factor**  $|f'(x)|$ .



In the study of reflexivity the factor was a constant  $a$ . Here, it will be a **velocity**.

# I. Free fall

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# Can an experiment prove the actual infinite?

**The tempting argument.** A body falls from position **0** to position **1**. Once the fall is over, it has gone through every point of  $[0, 1]$ , even non-constructible ones: the path is an actual infinity with the power of the continuum.

**Why it does not work.**

- **Circularity:** “every point is reached” is not observed; it follows from the *model*  $x : [0, T] \rightarrow \mathbb{R}$  continuous (intermediate value theorem).
- **Finite precision:** an experiment yields finitely many (time, position) pairs, each known up to an error.
- **Zeno:** that the body arrives shows the fall ends, not that its stages form an actual totality of cardinal  $2^{\aleph_0}$ .

Peano's axioms are not verified by the grid either (its last grain has no successor), and the continuum hypothesis is independent of ZFC (Gödel 1940, Cohen 1963): no experiment can decide it.

# Continuum and grid are indiscernible

## **Hypothesis 2** (Observations)

An observation is a position measurement at a standard time, with a standard precision  $\delta > 0$ .

Let  $x : [0, T] \rightarrow [0, 1]$  be a standard continuous path. Cut time into  $2^{kn_\infty}$  grains,  $t_m = m T \varepsilon$ , and set  $x_m = x(t_m)$ .

## **Proposition 3** (Indiscernibility)

For every standard time  $t$ , if  $t_m$  is the nearest junction,  $\text{st}(x_m) = x(t)$ . No observation distinguishes the continuous path from the grid path.

## Consequence

The experiment does not prove Cantor's continuum; it shows that continuum and hyperfinite grid account for the **same observations**. Not a proof of the actual infinite, but an argument of **physical admissibility** for grains.

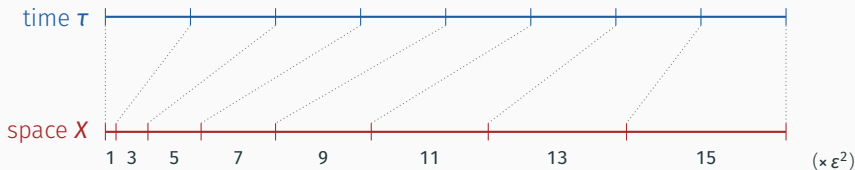
# Time grains, space grains

Free fall from rest:  $x = \frac{1}{2}gt^2$ . With  $\tau = t/T$  and  $X = x/L$  (where  $L = \frac{1}{2}gT^2$ ):

$$X(\tau) = \tau^2, \quad \tau \in [0, 1].$$

Uniform time grains  $\tau_m = m\varepsilon$  produce the space grains

$$G_m = [\tau_m^2, \tau_{m+1}^2[ = [m^2\varepsilon^2, (m+1)^2\varepsilon^2[.$$



# Galileo's odd-number law, exactly

## Theorem 4 (Odd-number law)

1. The endpoints of the space grains  $G_m$  are junctions of the homogeneous grid of resolution  $-2kn_\infty$  (step  $\varepsilon^2$ ).
2.  $G_m$  is the union of exactly  $2m + 1$  consecutive fine grains.
3. The first  $M$  time grains cover exactly  $\sum_{m < M} (2m + 1) = M^2$  fine grains.

Proof:  $m^2 \varepsilon^2 = m^2 2^{-2kn_\infty}$ , and  $(m + 1)^2 - m^2 = 2m + 1$ .

Distances covered in equal successive times are as  $1 : 3 : 5 : 7 : \dots$   
(Galileo, *Discorsi*, 1638)

## Exact, not approximate

In grain theory this law is not an approximation: it is **exact** at the level of grains, and it is a **count of hyperintegers**.

# Velocity as stratification factor

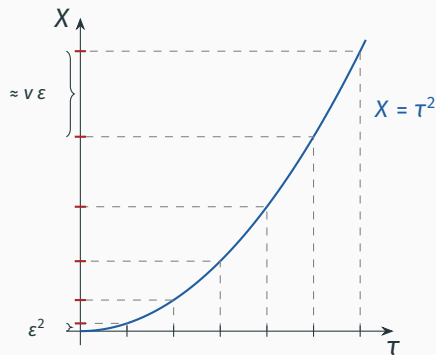
## Proposition 5 (Thickness of space grains)

$$(2m + 1) \varepsilon^2 = v(\tau_m) \varepsilon + \varepsilon^2, \quad v(\tau) = X'(\tau) = 2\tau.$$

To first order, this is  $|X'(\tau_m)| \varepsilon$ : the **velocity** is the stratification factor.

## Remark 6 (Doubly infinitesimal)

$G_0$  has thickness  $\varepsilon^2$ , infinitely small compared with a time grain  $\varepsilon$ ; the last one has thickness  $\approx 2\varepsilon$ . The ratio  $2^{kn_\infty+1} - 1$  is unlimited.



# Scale invariance: compatibility with refinement

## Proposition 7 (Refinement)

At resolution  $-N$  (any integer or hyperinteger  $N$ ),  $G_m^N$  contains  $2m + 1$  grains of resolution  $-2N$ . At resolution  $-(N + 1)$ , time grain  $m$  splits into  $2m$  and  $2m + 1$ , whose space grains contain  $4m + 1$  and  $4m + 3$  grains of resolution  $-(2N + 2)$ , and

$$(4m + 1) + (4m + 3) = 4(2m + 1).$$

$G_1^N$ : 3 grains of res.  $-2N$



12 grains of res.  $-(2N + 2)$



Here  $m = 1$ :  $5 + 7 = 12 = 4 \times 3$ . The law at one level follows exactly from the law at the next: the whole tower is coherent.

# Dilation invariance, not gauge invariance

## Proposition 8 (Dilations)

The parabola  $X = \tau^2$  is invariant under the anisotropic dilations  $D_\lambda : (\tau, X) \mapsto (\lambda\tau, \lambda^2 X)$ . For  $\lambda = 2^{-j}$ ,  $D_\lambda$  maps the grid of resolution  $-N$  onto the first  $2^N$  grains of the grid of resolution  $-(N+j)$ .

- Time has weight 1, space weight 2: hence space at resolution  $-2N$ .
- In physical units,  $(t, x) \mapsto (\lambda t, \lambda^2 x)$  leaves  $x = \frac{1}{2}gt^2$  invariant.

## Remark 9 (Not Weyl's gauge)

This is a **global** scale invariance: the resolution changes everywhere at once. Weyl's gauge invariance (1918) is a **local** change of scale,  $g \mapsto e^{2\lambda(x)}g$ , with a connection. Analogy only: the velocity  $\mathbf{v}(\tau)$  acts as a local scale factor, like Weyl's conformal factor.

# Power laws

## Proposition 10 (Power laws)

For  $X(\tau) = \tau^p$  ( $p \geq 1$  integer), at resolution  $-N$ :

- the endpoints of  $G_m^N$  are junctions of resolution  $-pN$ ;
- $G_m^N$  contains exactly  $(m+1)^p - m^p$  fine grains;
- the law is compatible with refinement and invariant under  $(\tau, X) \mapsto (\lambda\tau, \lambda^p X)$ .

| Law                 | Fine grains in $G_m$ | Sequence                         |
|---------------------|----------------------|----------------------------------|
| $p = 1$ (uniform)   | 1                    | 1, 1, 1, ... (no stratification) |
| $p = 2$ (free fall) | $2m + 1$             | 1, 3, 5, 7, ...                  |
| $p = 3$             | $3m^2 + 3m + 1$      | 1, 7, 19, 37, 61, ...            |

With friction,  $x(t) \propto \ln \cosh(t/t_c)$  is not homogeneous: the invariance is lost. Grains make scale invariance **exact and combinatorial**, but they **inherit** it from the law; they do not create it.

# Homogeneous counting measures mean velocity

## Proposition 11 (Mean velocity)

Let  $[a, b] \subset [0, 1]$  be a standard interval of positions. Let  $N_X$  be the number of homogeneous space grains of resolution  $-kn_\infty$  in  $[a, b]$ , and  $N_\tau$  the number of time grains spent in  $[a, b]$ . Then

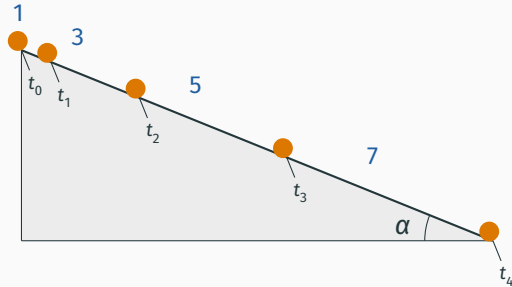
$$\text{st}\left(\frac{N_X}{N_\tau}\right) = \frac{b - a}{\sqrt{b} - \sqrt{a}} = \text{mean velocity on } [a, b].$$

- Start of the fall: the ratio is  $\sqrt{b} \rightarrow 0$ ; many time grains, few space grains.
- Whole fall:  $\text{st}(N_X/N_\tau) = 1 = \int_0^1 X'(\tau) d\tau$ .

In the reflexivity study, the ratio was the constant  $1/a$ ;  
here it becomes a **variable mean velocity**, and global counting involves  $\int |f'|$ .

# Galileo's experiment: the inclined plane

- Free fall was too fast to time in 1600.
- Galileo rolls a bronze ball in a polished groove on an inclined beam; time is measured by **weighing water**.
- Distances are as the squares of the times.
- The slope “**dilutes**” gravity: slower motion, same law.
- Reproduced by Settle (*Science*, 1961).



equal times  $t_0, \dots, t_4$ ; distances as 1 : 3 : 5 : 7 (to scale)

## The inclined plane: same theorem, any slope

Dynamics. Along the plane,  $s(t) = \frac{1}{2}at^2$  with

$$a = \frac{g \sin \alpha}{1 + I/(mr^2)} \quad \left( a = \frac{5}{7} g \sin \alpha \text{ for a solid ball on a plane} \right).$$

### Proposition 12 (Independence from the slope)

With  $\tau = t/T$  and  $S = s/L$ :  $S(\tau) = \tau^2$ . The odd-number law, its refinement compatibility and the velocity factor hold **identically**, for any slope  $\alpha$ , mass, moment of inertia or groove shape.

### Remark 13 (The slope realizes the dilation)

$s = \frac{1}{2}at^2$  is invariant under  $(t, a) \mapsto (\lambda t, a/\lambda^2)$ . Tilting the plane dilates time by  $\lambda = 1/\sqrt{\sin \alpha}$ : Galileo's "dilution" of gravity **lengthens every time grain** without changing the odd-number law.

# Four grids at once, and the reach of the experiment

## Corollary 14 (Four simultaneous grids)

For the rolling ball, four quantities are proportional to  $t^2$ : the distance  $s$  along the plane, the horizontal displacement  $s \cos \alpha$ , the vertical drop  $s \sin \alpha$ , and the rotation angle  $\varphi = s/r$ . One uniform time grid produces four stratified grids, each with  $2m + 1$  fine grains in its  $m$ -th grain.

## What the experiment verifies, and what grains extend

- Galileo verifies the law at a few **standard** resolutions, with finite precision.
- He cannot verify it at unlimited resolutions.
- Grains extend the observed law **coherently** to all resolutions (refinement), and promotion gives back  $s = \frac{1}{2}at^2$ .

The experiment fixes the law at accessible scales; the grain extends it coherently; promotion gives its continuous form.

# The continuous path as a promotion

## Theorem 15 (Promotion of the fall)

1. For every standard  $\tau$ , if  $\tau_m$  is the nearest junction,  $\text{st}(\tau_m^2) = \tau^2$ : the grid path promotes to  $X = \tau^2$ .
2. Velocity is the promotion of the rate of one grain:

$$\text{st}\left(\frac{\tau_{m+1}^2 - \tau_m^2}{\varepsilon}\right) = \text{st}((2m+1)\varepsilon) = 2\tau = v(\tau).$$

Promotion **erases** the thickness of the grains;  
it **keeps** their ratio, which becomes the derivative.

## Part I in one line

Odd numbers on the grid  $\xrightarrow{\text{st}}$  a differential law on the continuum.

## II. The brachistochrone in layers of grains

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# The problem and Bernoulli's optical solution

**Brachistochrone** (Johann Bernoulli, 1696): the curve along which a body starting from rest descends from **A** to **B** in **minimal time**. Answer: an arc of **cycloid**.

**Bernoulli's solution** (*Acta Eruditorum*, 1697):

1. by energy conservation, the speed depends only on the drop:  $v = \sqrt{2gy}$ ;
2. cut the medium into thin horizontal **layers** of constant speed;
3. the fastest path is a light ray: at each interface, **Snell's law**  $\sin \varphi / v = \text{const}$  (Fermat);
4. let the thickness of the layers tend to zero.

## In grain theory

Steps 2 and 4 are a **stratification followed by a limit**: the layers become grains of the vertical axis; the limit becomes a **promotion**.

# Setting

Axis  $y$  pointing **down**, start at  $A = (0, 0)$ , speed  $v(y) = \sqrt{2gy}$ . Descending paths  $x = x(y)$ ,  $y \in [0, Y]$ , with travel time

$$T[x] = \int_0^Y \frac{\sqrt{1 + x'(y)^2}}{\sqrt{2gy}} dy.$$

The cycloid of radius  $R$  through  $A$ :

$$x = R(\theta - \sin \theta), \quad y = R(1 - \cos \theta), \quad \frac{dx_R}{dy} = \tan \frac{\theta}{2} = \sqrt{\frac{y}{2R - y}}.$$

We assume  $B$  is reached before the lowest point ( $\theta_B < \pi$ ).

Grid.  $N = kn_{\infty}$ ,  $\varepsilon = Y 2^{-N}$ , and  $[0, Y]$  is cut into  $M = 2^N$  layers  $[j\varepsilon, (j+1)\varepsilon]$ .

## A medium in layers of grains

### Definition 16 (Layered medium)

In layer  $j$ , pick a witness height  $\eta_j \in [j\varepsilon, (j+1)\varepsilon]$  (by default the midpoint) and give the body the constant speed  $v_j = \sqrt{2g\eta_j}$ .

### Definition 17 (Grain path)

A broken line with, in layer  $j$ , a segment of slope  $u_j = \tan \varphi_j$  ( $\varphi_j$ : angle with the vertical). Horizontal displacement and travel time:

$$X(u) = \sum_{j=0}^{M-1} \varepsilon u_j, \quad T(u) = \sum_{j=0}^{M-1} \frac{\varepsilon \sqrt{1 + u_j^2}}{v_j}.$$

In layer  $j$  the segment has length  $\varepsilon \sqrt{1 + u_j^2}$ , travelled at speed  $v_j$ .

# Snell's law in grains

## Theorem 18 (Snell's law in grains)

For  $X_B \geq 0$ , among grain paths with  $X(u) = X_B$  there is **exactly one** that minimizes  $T(u)$ . It is characterized by

$$\frac{\sin \varphi_j}{v_j} = \lambda \quad \text{for all } j,$$

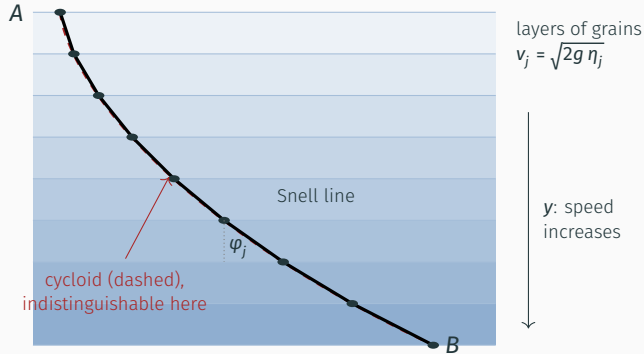
where  $\lambda \in [0, 1 / \max_j v_j[$  is the unique constant with  $X(u) = X_B$ .

**Proof.** For standard  $M$ :  $u \mapsto \sqrt{1 + u^2}$  is strictly convex, so  $T$  is strictly convex and coercive; on the hyperplane  $\{X = X_B\}$  it has a unique minimum, given by Lagrange:  $\partial T / \partial u_j = \varepsilon u_j / (v_j \sqrt{1 + u_j^2}) = \lambda \varepsilon$ , i.e.  $\sin \varphi_j = \lambda v_j$ . The map  $\lambda \mapsto X$  is continuous and strictly increasing onto  $[0, \infty[$ . The statement is first-order: by **transfer** it holds for  $M = 2^{kn_\infty}$ .

## Fermat proved, not invoked

What Bernoulli obtained by optical analogy is, in each grid, an **exact theorem**: Fermat's principle is **proved**, not invoked.

# Bernoulli's construction in grains



With only 8 layers, the fastest broken line is already almost the cycloid. At an unlimited resolution, its promotion is the cycloid.

# From the Snell line to the cycloid

Take  $2R > Y$  and  $\lambda_R = 1/(2\sqrt{gR})$ , so that  $\lambda_R v = 1$  exactly at the lowest point  $y = 2R$ .

## Lemma 19

If  $\sin \varphi = \lambda_R \sqrt{2g\eta}$ , then  $\tan \varphi = \sqrt{\eta/(2R - \eta)}$ : the slope of the cycloid at the witness height.

## Theorem 20 (Promotion of the Snell line)

Let  $x_j = \sum_{i < j} \varepsilon u_i$  be the vertices of the Snell line.

1. For every standard  $y$ , if  $j\varepsilon$  is the nearest junction,  $\text{st}(x_j) = x_R(y)$ : the Snell line promotes to the cycloid.
2. Its travel time promotes to that of the cycloid:

$$\text{st}(T(u)) = \sqrt{R/g} \theta_Y.$$

$x_j$  is a Riemann sum with infinitesimal step, so its standard part is  $\int_0^y \sqrt{\eta/(2R - \eta)} d\eta = x_R(y)$ .

# The travel time: an improper integral

The time is a Riemann sum  $T(u) = \sum_j \varepsilon h(\eta_j)$  with

$$h(\eta) = \frac{\sqrt{1 + \tan^2 \varphi}}{\sqrt{2g\eta}} = \sqrt{\frac{R}{g\eta(2R - \eta)}},$$

unbounded near 0, where the speed vanishes.

**Split the sum.** For a standard  $\delta > 0$ :

- layers with  $\eta_j \geq \delta$ : Riemann sum of a continuous function, standard part  $\int_\delta^Y h$ ;
- layers with  $\eta_j < \delta$ : bounded by  $C \sum \varepsilon / \sqrt{\eta_j} \leq C' \sqrt{\delta}$ .

Hence  $|\text{st}(T(u)) - \int_0^Y h| \leq C' \sqrt{\delta}$  for all standard  $\delta$ , so  $\text{st}(T(u)) = \int_0^Y h$ .

**Change of variable.**  $y = R(1 - \cos \theta)$  gives  $h dy = \sqrt{R/g} d\theta$ , hence  $\int_0^Y h = \sqrt{R/g} \theta_Y$ .

## Four remarks

### Remark 21 (One level suffices)

Bernoulli lets the layers shrink to zero; here **one unlimited resolution** suffices, as for nested segments in the study of Dedekind cuts.

### Remark 22 (Witness heights)

Any choice of  $\eta_j$  in the layers gives the same promotion: the broken line changes, the cycloid does not.

### Remark 23 (Imposed endpoint)

For a standard  $B$  with  $\theta_B < \pi$ ,  $\lambda$  depends on the resolution, but  $\text{st}(\lambda) = \lambda_R$  for the cycloid through  $B$ .

### Discrete optimum vs. continuous optimum

We prove optimality **among grain paths**, and that the promotion is the cycloid. Proving directly that “the promotion of a discrete optimum is a continuous optimum” needs more (e.g.  $\Gamma$ -convergence): **open**.

# Kinematics on the cycloid

## **Proposition 24** (Uniform parameter)

Along the cycloid, a body released at rest at **A** satisfies

$$\theta(t) = \sqrt{g/R} t.$$

A uniform time grid is a **uniform grid in  $\theta$** .

Proof:  $v^2 = 2R^2\dot{\theta}^2(1 - \cos \theta)$  and  $v^2 = 2gy = 2gR(1 - \cos \theta)$ , so  $\dot{\theta}^2 = g/R$ .

Arc length from **A**:

$$s(\theta) = 4R\left(1 - \cos \frac{\theta}{2}\right) = 8R \sin^2 \frac{\theta}{4} \approx \frac{1}{2}R \theta^2 = \frac{1}{2}gt^2 \quad \text{near } A.$$

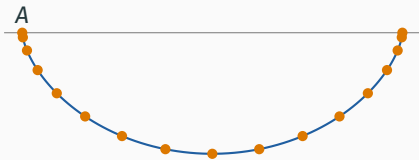
Near the start, it is **free fall**: the tangent at the cusp **A** is vertical.

# Galileo at the infinitesimal scale

## Theorem 25 (Infinitesimal odd-number law)

Let  $s$  be standard,  $C^2$  near 0, with  $s(0) = s'(0) = 0$  and  $s''(0) = a_0 \neq 0$  (start from rest). Let  $\Delta$  be an infinitesimal time step and  $s_m = s(m\Delta)$ . For every standard  $m$ ,

$$\text{st}\left(\frac{s_{m+1} - s_m}{s_1 - s_0}\right) = 2m + 1.$$



successive arcs at equal times:  
cycloid 1 : 2.96 : 4.81 : 6.47 : ...  
free fall 1 : 3 : 5 : 7 : ...

Every start from rest is Galilean at the scale of grains; the law is exact at all ranks only for  $t^2$  laws. On the cycloid, grain endpoints involve cosines: combinatorial exactness is lost.

# The tautochrone: an invariant count

## **Proposition 26** (Huygens, 1673)

A body released at rest at any  $\theta_0 \in [0, \pi[$  reaches the lowest point after

$$T_* = \pi\sqrt{R/g}, \quad \text{independent of } \theta_0.$$

Proof: with  $c = \cos(\theta/2)$ ,  $T_* = \sqrt{R/g} \int_0^{c_0} 2 \, dc / \sqrt{c_0^2 - c^2} = \pi\sqrt{R/g}$ .

## **Corollary 27** (Invariant count)

For a time grain of duration  $\Delta$ , the number of time grains needed to reach the bottom,  $N_* = \lfloor T_*/\Delta \rfloor$ , is the **same hyperinteger** for every starting point.

The depth is Huygens'; the corollary translates it. Combined with the previous theorem: all starts are Galilean, with different initial accelerations  $g \cos(\theta_0/2)$ , yet all arrive after the same number of grains. This is the grain form of the **harmonic** motion on the cycloid.

# Synthesis

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# Three motions compared

|                       | Free fall             | Inclined plane        | Cycloid                          |
|-----------------------|-----------------------|-----------------------|----------------------------------|
| Law                   | $x = \frac{1}{2}gt^2$ | $s = \frac{1}{2}at^2$ | $s = 8R \sin^2(\theta/4)$        |
| Space grain endpoints | dyadic junctions      | dyadic junctions      | cosines                          |
| Odd-number law        | exact, all ranks      | exact, all ranks      | standard ranks only              |
| Scale invariance      | global, exact         | global, exact         | infinitesimal only               |
| Stratification factor | $v = gt$              | $v = at$              | $v = 2\sqrt{gR} \sin(\theta/2)$  |
| Specific result       | mean velocity         | four grids            | Snell, tautochrone               |
| Promotion             | $X = \tau^2$          | $S = \tau^2$          | cycloid, $T = \sqrt{R/g} \theta$ |

## Common thesis

Grids carry **sharp, discrete** structure — odd numbers, exact optimality conditions, invariant counts. **Promotion** turns it into the differential laws and variational principles of classical mechanics.

## Open questions

1. **Discrete to continuous optimum:** does promotion commute with minimization? A  $\Gamma$ -convergence argument in grain language.
2. **Paths below the endpoint:** extend the layered construction to  $\theta_B > \pi$ , where the brachistochrone dips below  $B$  and climbs back.
3. **General motions:** for an arbitrary  $\mathbf{x}(t)$ , space grains are stratified by  $|\mathbf{x}'(t)|$  but no longer aligned with a homogeneous grid. What survives of the counting results?
4. **A local scale invariance:** a resolution varying from region to region, with a matching rule, would allow a serious look at the analogy with Weyl's gauge.

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