The Projective Möbius Band in the Grain Program

Decomposition, cuts, stratal information, and a DNA analogy

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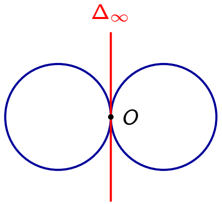
Overview

The symmetric flow

Notation (conflict resolved)

The two circles of the symmetric flow are C_{+kn_∞} and C'_{+kn_∞} — same resolution level $+kn_\infty$, the prime marking symmetry (not the level $\mathcal{G}_{-kn_\infty}$).

- Two circles tangent at O;
- the line at infinity Δ_∞ is the projective axis;
- $O_{+\infty} = O_{-\infty} = O_\infty$ identified under the turn.



$\mathbb{RP}^2$ as disk glued to a Möbius band

Decomposition

$$\mathbb{RP}^2 = D^2 \cup_{S^1} \text{Möbius}.$$

A tubular neighbourhood of the line at infinity Δ_∞ *is* a Möbius band; its core is Δ_∞, its boundary is the flow.

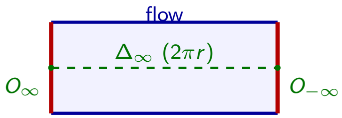
- The flow $C_{+kn_\infty} \cup C'_{+kn_\infty}$ is the *seam* circle separating the finite disk D^2 (side of O) from the Möbius band (side of Δ_∞).
- Two registers kept separate: *topological* (global shape) vs *stratal/metric* (measures near O).

The band: two perpendicular median lines

Before gluing, the construction is a rectangular band. **Two** median lines must not be confused:

- **core** Δ_∞: longitudinal median, length $2\pi r$, endpoints O_∞, $O_{-\infty}$;
- **gluing**: at the two *short ends* (with a turn), identifying $O_\infty \sim O_{-\infty}$.

Width ℓ is *chosen*; any $\ell > 0$ gives a Möbius band.



The transverse cut

Result

Cutting transversely (across Δ_∞) reopens the gluing and yields a flat orientable band with **four borders**:

- two long borders = the flow C_{+kn_∞}, C'_{+kn_∞};
- two transverse borders = the two *lips* of the cut.

Here Δ_∞ is the *median line*, not a border.

Two cuts, two results

Transverse: Δ_∞ stays median. **Longitudinal** (along Δ_∞): Δ_∞ doubles into a border; one obtains a band with 4 half-twists. The two must not be conflated.

Cutting theorems (topological, by invariants)

For R_n (band with n half-twists):

- **Orientable** $\Leftrightarrow$ n even (parity lemma);
- **# borders** $= 1$ if n odd, 2 if n even.

Median cut

Cutting R_n along its core gives, for n odd, *one* band with $2n + 2$ half-twists (e.g. $n = 1 \rightarrow$ one two-sided band, 720°, no longer a Möbius). Cutting at one-third gives *two* linked loops.

These are proved by topological invariants (orientability, border count), not by empirical folding — though the DNA-origami experiment confirms them.

Two invariants, not to be fused

A Möbius linking levels k and j (width $\ell_j = 2^{jn_\infty}$) carries *two* invariants of different natures:

Turn $\phi = \pi$

an *angle*, constant, independent of j, k; the signature of non-orientability.

Scale ratio $2^{(j-k)n_\infty}$

a *multiplicative factor* in $\mathbb{R}_{>0}$, not an angle.

Additive S^1 (turn) vs multiplicative $\mathbb{R}_{>0}$ (scale) — calling both "phase" would fuse them.

The information hierarchy, resolved by the logarithm

Grains of level j: $N = 2^{jn_\infty}$. Hartley information $\mathcal{I} = \log_2 N$ sends cardinals to exponents:

$$\mathcal{I}(C_{kn_\infty}) = kn_\infty \;<\; \mathcal{I}(C_{jn_\infty}) = jn_\infty \;<\; \mathcal{I}(\Delta_\infty) = \aleph_0 = \sup \mathcal{I}.$$

- First two: *hyperfinite* levels (same nature, comparable);
- last: cardinal $\aleph_0$, reached by *promotion*, not by finer counting.
- Without the log, 2^{kn_∞} (hyperfinite) vs $2^{\aleph_0}$ (continuum) are incomparable — the log makes the hierarchy legible.

Möbius and DNA: analogy, not equivalence

What is real (experiment)

Han–Pal–Liu–Yan (*Nature Nanotech.*, 2010) built a Möbius band from DNA origami ($\sim 50\,\mathrm{nm}$). Median cut $\to$ 720° band / two borders, confirmed. The parity criterion (Möbius $\Leftrightarrow$ odd twists) is EM-verified.

- The a-priori cutting theorems are *illustrated* experimentally.
- **Limit**: a naive strand junction is chemically impossible for natural DNA (antiparallel strands); the Möbius topology is realized by *origami*, not by a strand cross-junction.
- Bridge to the grain (which stratal invariant $\leftrightarrow$ linking number?) *remains to build*.

Summary

- **Flow**: $C_{+kn_\infty}, C'_{+kn_\infty}$ at one level; $\mathbb{RP}^2 = D^2 \cup_{S^1}$ Möbius, core Δ_∞.
- **Geometry**: core Δ_∞ (long, $2\pi r$) vs gluing at short ends ($O_\infty \sim O_{-\infty}$); width ℓ chosen.
- **Cuts**: transverse (4 borders, Δ_∞ median) vs longitudinal (Δ_∞ becomes border, 4 half-twists); cutting theorems by invariants.
- **Invariants**: turn $\phi = \pi$ (angle) vs scale $2^{(j-k)n_\infty}$ (factor); information $kn_\infty < jn_\infty < \aleph_0$ via the log.
- **DNA**: experimental analogy (origami), grain bridge open.

Method

Topology proved by invariants; stratal claims carried by numbers; analogies labelled as such. Where a measurable quantity is missing, the claim stays a suggestion.

Thank you

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