A Family of Nested Circular Loops

Homeomorphisms, Viète's formula and the topological status of the circle

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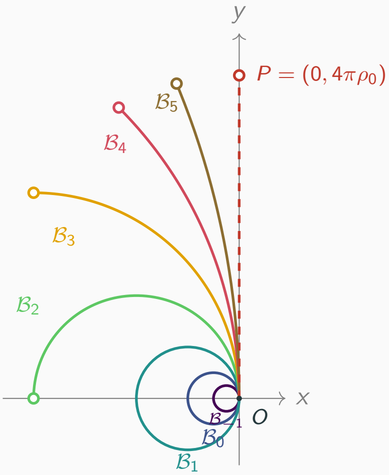
Based on the paper of 1 May 2004

Outline

1. The family $(\mathcal{B}_k)_{k\in\mathbb{Z}}$
2. The homeomorphism ψ
3. The variant ψ^* and the topological status of the circle
4. A geometric proof of Viète's formula
5. Asymptotic behaviour

The idea in one picture

- A one-parameter family $(\mathcal{B}_k)_{k\in\mathbb{Z}}$ of curves drawn on circles through the origin O.
- All have the same length $4\pi\rho_0$: arcs, a circle, or loops covered many times.
- A simple map ψ, built on the radial factor $\cos\theta$, sends $\mathcal{B}_k$ onto $\mathcal{B}_{k-1}$.
- Consequences:
  1. the circle as the image of a *half-open* half-circle;
  2. a geometric proof of Viète's product (1593);
  3. two limits, $k \to +\infty$ and $k \to -\infty$.



The family $(\mathcal{B}_k)_{k\in\mathbb{Z}}$

Definition

Orthonormal frame $\mathcal{R} = (O, \vec{\imath}, \vec{\jmath})$, fixed $\rho_0 > 0$.

Definition 1
For $k \in \mathbb{Z}$, C_k is the circle of centre $O_k(-2^k \rho_0, 0)$ and radius $\rho_k = 2^k \rho_0$. The curve $\mathcal{B}_k$ is C_k travelled counter-clockwise *from* O, through a total angle

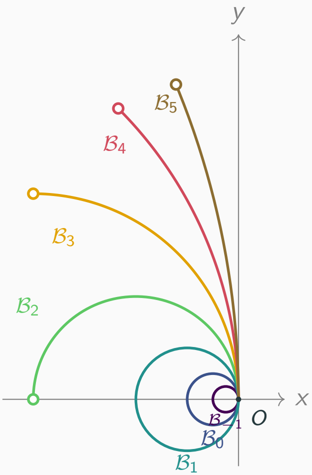
$$\theta_k^{\text{tot}} = 2^{-k+1} \cdot 2\pi.$$

Its end point S_k is excluded by convention.

- Every C_k passes through O and is tangent there to the y-axis.
- Parametrisation by the angle at the centre: $T(\theta) = O_k + \rho_k(\cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath})$, $\theta \in [0,\, 2\pi \cdot 2^{1-k})$.

Three regimes

k	turns $N_k = 2^{1-k}$	shape of $\mathcal{B}_k$
$k > 1$	< 1	arc of circle
	$k = 2$: $\frac{1}{2}$	half circle
	$k = 3$: $\frac{1}{4}$	quarter circle
$k = 1$	1	the circle C_1 once
$k \leq 0$	≥ 2	multi-loop
	$k = 0$: 2	C_0 twice
	$k = -n$: 2^{n+1}	C_{-n}, 2^{n+1} times

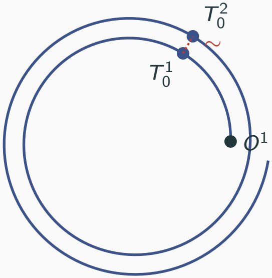


Covering index and geometric classes

A point of a multi-loop may be visited several times: we write T_k^β, $\beta \in \{1, \dots, 2^{-k+1}\}$, where β says *which passage* is meant. The start point is O^1.

Definition 2
$T_k^\beta \sim T_k^{\beta'}$ iff the two points coincide in the plane.

- $\sim$ is an equivalence relation; class of T_k^β: $\widetilde{T}_k$.
- $k > 0$: every class is a singleton.
- $k \leq 0$: every class has exactly 2^{-k+1} elements.



$\mathcal{B}_0$: C_0 covered twice
(drawn slightly "lifted")

A common length

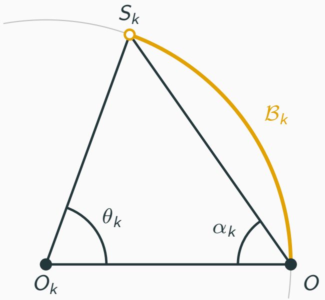
Proposition (Common length)
For all $k \in \mathbb{Z}$, the arc length of $\mathcal{B}_k$ is $4\pi\rho_0$.

Proof.
Length

$$= \theta_k^{\text{tot}} \cdot \rho_k = 2\pi\, 2^{-k+1} \cdot 2^k \rho_0 = 4\pi\rho_0. \qquad \square$$

Remark (Angles, arc case $k > 1$)
At the centre and at the origin:

$$\theta_k = \widehat{OO_kS_k} = \frac{\pi}{2^{k-2}}, \qquad \alpha_k = \widehat{O_kOS_k} = \frac{\pi}{2}\left(1 - \frac{1}{2^{k-2}}\right).$$



$\theta_k \to 0$ and $\alpha_k \to \pi/2$ as $k \to +\infty$ (triangle O_kOS_k is isosceles).

The homeomorphism ψ

Construction

For $\alpha > 0$, let $\alpha\{C_k\}$ be C_k travelled α times from O (end point excluded). A point T_k^β is located by $\theta_k^\beta = \widehat{OO_k T_k^\beta} \in [0, 2\pi\alpha)$.

Definition 5

$$\psi : \ \alpha\{C_k\} \longrightarrow 2\alpha\{C_{k-1}\}, \qquad T_k^\beta \longmapsto T_{k-1}^{\beta'} \quad \text{with} \quad \overrightarrow{O_k T_{k-1}^{\beta'}} = \cos(\theta_k^\beta)\,\overrightarrow{O_k T_k^\beta}.$$

Lemma 6
$T_{k-1}^{\beta'}$ lies on C_{k-1} and its angle measured from O_{k-1} is

$$\theta_{k-1}^{\beta'} = 2\,\theta_k^\beta.$$

Proof of Lemma 6

Write $\vec{e}_r^{\,k} = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath}$ with $\theta = \theta_k^{\beta}$. Then

$$\begin{aligned}\overrightarrow{O_k T_{k-1}^{\beta'}} &= \rho_k \cos\theta\ \vec{e}_r^{\,k} = \rho_k\big(\cos^2\theta\,\vec{\imath} + \cos\theta\sin\theta\,\vec{\jmath}\big)\\ &= \rho_{k-1}\big((1+\cos 2\theta)\,\vec{\imath} + \sin 2\theta\,\vec{\jmath}\big) \qquad (\rho_k = 2\rho_{k-1},\ \text{duplication}).\end{aligned}$$

Since $\overrightarrow{O_{k-1}O_k} = -\rho_{k-1}\vec{\imath}$,

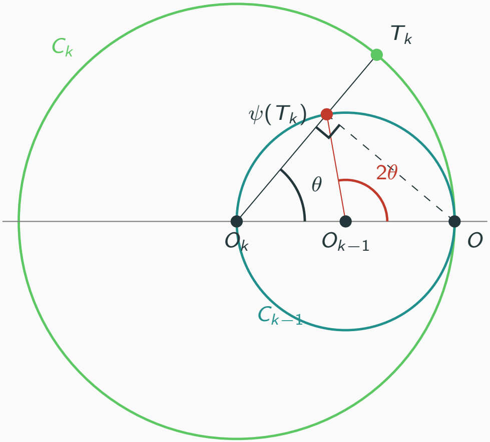
$$\boxed{\overrightarrow{O_{k-1} T_{k-1}^{\beta'}} = \rho_{k-1}\big(\cos 2\theta\,\vec{\imath} + \sin 2\theta\,\vec{\jmath}\big)}$$

so the image is at distance ρ_{k-1} from O_{k-1}, with angle 2θ. $\square$

Remark (Remark 9)
$T_{k-1}^{\beta'}$ is the second intersection of the line $(O_k T_k^{\beta})$ with C_{k-1} — the first one being O_k itself, since $O_k \in C_{k-1}$.

Two classical facts make it transparent:

- $[O_k O]$ is a *diameter* of C_{k-1}: by Thales, the image is the orthogonal projection of O onto $(O_k T)$, at distance $\rho_k \cos\theta$ from O_k.
- Inscribed angle theorem: the angle θ at $O_k \in C_{k-1}$ becomes the central angle 2θ at O_{k-1}.



Bijection and continuity

Proposition (Prop. 7)
ψ is a bijection of $\alpha\{C_k\}$ onto $2\alpha\{C_{k-1}\}$.

In parameters ψ is $\theta \mapsto 2\theta$, an increasing bijection $[0, 2\pi\alpha) \to [0, 4\pi\alpha)$.

Proposition (Prop. 8)
ψ is a homeomorphism.

ψ is conjugate to the linear map $\theta \mapsto 2\theta$, continuous with continuous inverse; the angular parametrisations identify the indexed loops with their parameter intervals.

$$\begin{tikzcd}
{[0, 2\pi\alpha)} \arrow[r, "\theta\mapsto 2\theta"] \arrow[d, "\text{param.}"'] & {[0, 4\pi\alpha)} \arrow[d, "\text{param.}"] \\
\alpha\{C_k\} \arrow[r, "\psi"'] & 2\alpha\{C_{k-1}\}
\end{tikzcd}$$

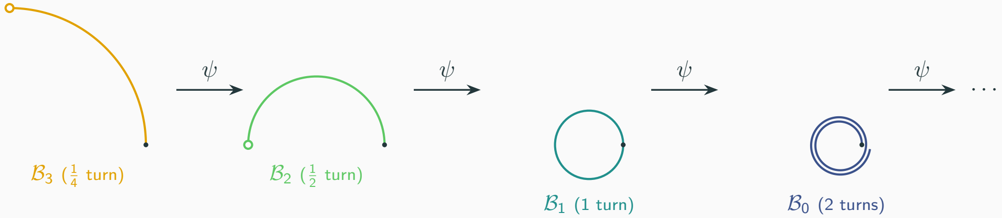
Arc length is preserved:

$$\rho_{k-1} \cdot 2\theta = \rho_k \theta.$$

Action on the whole family

Since $\mathcal{B}_k = 2^{-k+1}\{C_k\}$, Definition 5 gives directly:

Corollary 10
$\psi(\mathcal{B}_k) = \mathcal{B}_{k-1}, \quad \psi^{-1}(\mathcal{B}_k) = \mathcal{B}_{k+1}, \quad$ and $\psi^{\circ k}(\mathcal{B}_j) = \mathcal{B}_{j-k}$ for all $k, j \in \mathbb{Z}$.



ψ doubles the number of turns and halves the radius: the length $4\pi\rho_0$ is invariant.

The variant ψ^* and the topological status of the circle

From a half circle to a whole circle

Take $\alpha = \frac{1}{2}$ on C_1: the domain of ψ is the upper half circle $\frac{1}{2}\{C_1\} = (O^1; \bar{S})$, where S is diametrically opposite to O (excluded).

Definition 11
$\psi^* : (\bar{O}^1; S] \to \{C_0\}, \qquad \psi^*(T) = \psi(T) \text{ if } T \in (\bar{O}^1; \bar{S}), \qquad \psi^*(S) = O^2,$

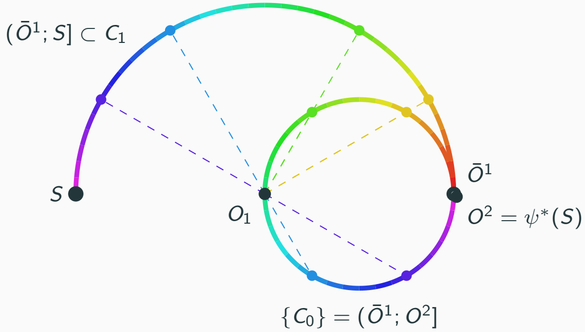
where O^2 is the origin with covering index 2: the return point after one full turn on C_0.

- Domain: half-open arc — *origin excluded, end point included*.
- Target: the whole circle seen as $(\bar{O}^1; O^2]$.

Theorem 12
$\psi^* : (\bar{O}^1; S] \to \{C_0\}$ is a homeomorphism.

Proof.

- Bijective: ψ is bijective on the interior, and one point is added on each side.
- Continuity inside: Prop. 8.
- At S: as $\theta \to \pi^-$, $2\theta \to 2\pi^-$, so the image tends to $O^2 = \psi^*(S)$. Same argument for $(\psi^*)^{-1}$ at O^2. $\square$



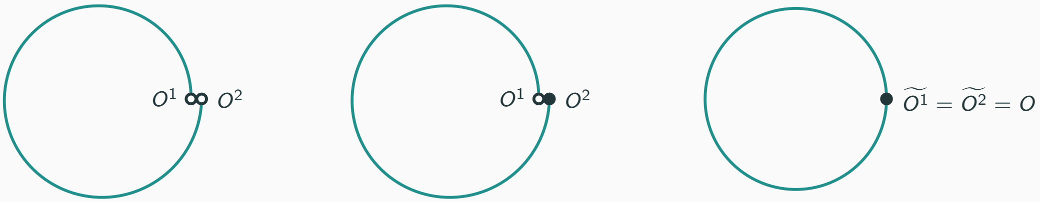
Colours: T at angle θ on the half circle and $\psi^*(T)$ at angle 2θ share a colour.

Topological consequence

Corollary 13
The circle $\{C_0\}$, identified with $(\bar{O}^1; O^2]$, is homeomorphic to the half-open interval $(0, \pi]$ through the angular parametrisation.

- Classical in general topology: $S^1 \cong [0, 2\pi]/(0 \sim 2\pi)$.
- Here it has a direct geometric meaning: stretching a half circle continuously yields a whole circle, provided the final end point is included — it is then identified with the origin through $\sim$.

Three viewpoints on the circle (Remark 14)



open

$(\bar{O}^1; \bar{O}^2) = \psi((\bar{O}^1; \bar{S}))$

half-open

$(\bar{O}^1; O^2]$ (parametrised)

closed (compact)

classes: the orbit is sealed

- The three viewpoints are consistent: they differ by whether one reasons on *indexed points* (with covering index) or on their *geometric classes*.

A precision on the topology used

- As a subset of the plane, the circle is compact; $(0, \pi]$ is not. Removing an interior point disconnects $(0, \pi]$ but not the circle. So the geometric circle $C_0 \subset \mathbb{R}^2$ is *not* homeomorphic to $(0, \pi]$.
- What ψ^* gives with the plane topology: a continuous bijection $(0, \pi] \to C_0$, whose inverse jumps at O (points near O^1 and near O^2 are close in the plane, far in the parameter).
- Theorem 12 and Corollary 13 hold in the indexed space $(\bar{O}^1; O^2]$, whose topology is the one of the parameter — the "half-open" viewpoint of Remark 14.
- Passing to classes $(\widetilde{O^1} = \widetilde{O^2})$ is exactly the quotient $[0, 2\pi]/(0 \sim 2\pi) \cong S^1$: this is where compactness appears.

A geometric proof of Viète's formula

Viète's product (1593)

Theorem 15 (Viète)

$$\frac{2}{\pi} = \cos\frac{\pi}{4}\cdot\cos\frac{\pi}{8}\cdot\cos\frac{\pi}{16}\cdots = \sqrt{\tfrac{1}{2}}\cdot\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}\cdot\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}}\cdots$$

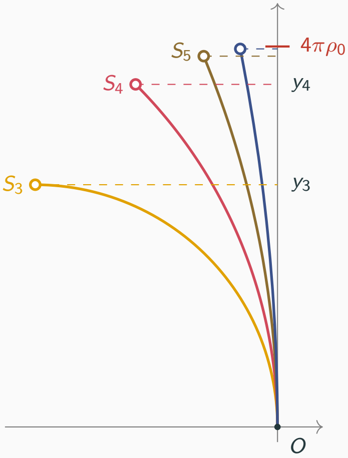
- Idea: follow the ordinate y_k of the end point S_k of the arcs $\mathcal{B}_k$ $(k \geq 3)$.
- All arcs have length $4\pi\rho_0$ and straighten towards the vertical: $y_k \nearrow 4\pi\rho_0$.
- Duplication formula $\Rightarrow$ each step multiplies y_k by $1/\cos$.

The ordinates y_k

Translating the frame to O_k:

$$y_k = \rho_k \sin\theta_k = 2^k \rho_0 \sin\frac{\pi}{2^{k-2}}.$$

k	y_k/ρ_0
3	8
4	$8\sqrt{2} \approx 11.314$
5	12.246
6	12.486
7	12.546
∞	$4\pi \approx 12.566$



Recurrence and telescoping

With $\sin 2x = 2\sin x\cos x$ applied to $x = \pi/2^{k-1}$:

$$y_k = 2^k\rho_0 \cdot 2\sin\tfrac{\pi}{2^{k-1}}\cos\tfrac{\pi}{2^{k-1}} = y_{k+1}\cos\tfrac{\pi}{2^{k-1}} \quad\Longrightarrow\quad y_{k+1} = \frac{y_k}{\cos(\pi/2^{k-1})} \quad (k \geq 3).$$

Telescoping from $y_3 = 8\rho_0$: $\; y_{p+1} = 8\rho_0 \prod_{j=3}^{p} \dfrac{1}{\cos(\pi/2^{j-1})}$.

Since $y_k \to 4\pi\rho_0$ (length $4\pi\rho_0$ and $\alpha_k \to \pi/2$; directly: $2^k\sin\frac{\pi}{2^{k-2}} \to 4\pi$):

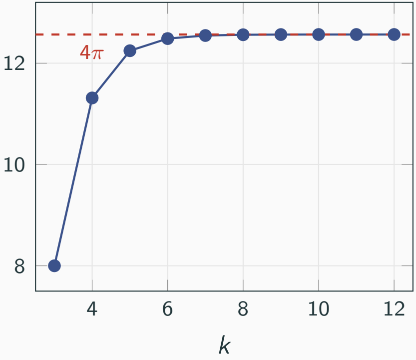
$$\prod_{j=3}^{\infty}\frac{1}{\cos(\pi/2^{j-1})} = \frac{\pi}{2}, \qquad \text{i.e.} \qquad \boxed{\prod_{n=2}^{\infty}\cos\frac{\pi}{2^n} = \frac{2}{\pi}}.$$

Nested radicals: $u_1 = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $u_{n+1} = \sqrt{(1+u_n)/2}$ (half-angle formula).

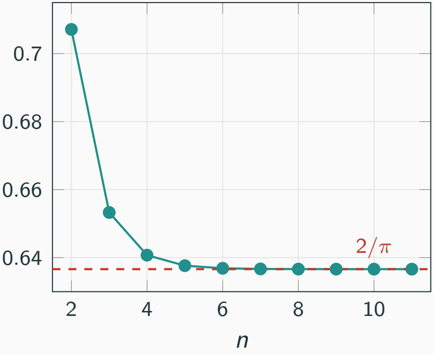
Note: the paper writes $y_3 = 4\rho_0$ and $\prod 1/\cos = \pi$; since $S_3 = (-8\rho_0, 8\rho_0)$, the values above are the consistent ones, and the final formula is unchanged.

Convergence, numerically

ordinates y_k/ρ_0



partial products $\prod_{m=2}^{n} \cos(\pi/2^m)$



Error on y_k decreases by a factor ≈ 4 at each step: $4\pi\rho_0 - y_k \sim \frac{(4\pi)^3}{6}\,\frac{\rho_0}{4^k}$.

Asymptotic behaviour

Limit $k \to +\infty$: straightening

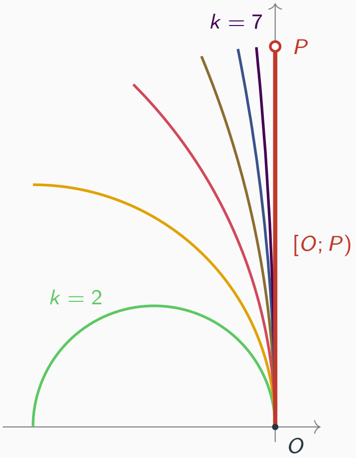
- Arcs flatten: $\theta_k \to 0$, $\alpha_k \to \pi/2$, length $4\pi\rho_0$. The paper conjectures

$$\lim_{k\to+\infty} \mathcal{B}_k = [O; P), \qquad P = (0, 2\pi\rho_1) = (0, 4\pi\rho_0).$$

- **Complement.** Arc-length parametrisation:

$$\gamma_k(s) = \left(\rho_k(\cos \tfrac{s}{\rho_k} - 1),\ \rho_k \sin \tfrac{s}{\rho_k}\right).$$

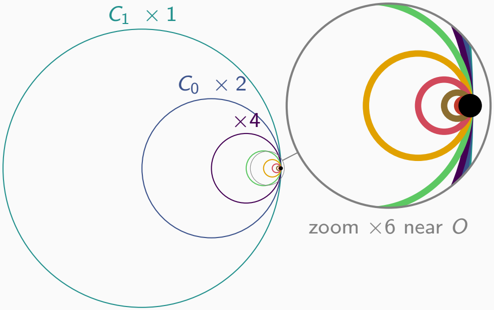
Since $|x| \le \frac{s^2}{2\rho_k}$ and $|y - s| \le \frac{s^3}{6\rho_k^2}$: $\sup_s |\gamma_k(s) - (0, s)| = O(2^{-k}) \to 0$. The conjecture holds, uniformly (and in Hausdorff distance).



Limit $k \to -\infty$: loops that shrink and multiply

With $k = -n$: $\mathcal{B}_{-n} = 2^{n+1}\{C_{-n}\}$, $\rho_{-n} = \rho_0/2^n \to 0$, $N_{-n} = 2^{n+1} \to +\infty$ turns.

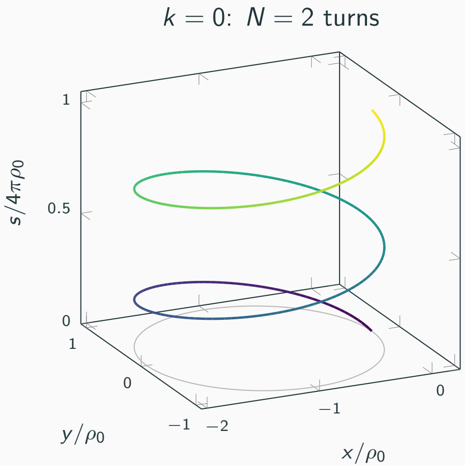
- Centres $O_k \to O$: the pointwise limit is O.
- Yet the length is invariant: $N_k \cdot 2\pi\rho_k = 4\pi\rho_0$.
- Self-similar: the homothety $(O, \frac{1}{2})$ maps C_k onto C_{k-1}; each zoom looks the same, but the multiplicity doubles.



A loop covered 2^{n_∞} times: lifting the covering index

To see the overlapping passages, lift each point to its arc length: $\Gamma_k(s) = \big(\gamma_k(s),\, s\big)$, $s \in [0, 4\pi\rho_0)$.

- The shadow on the plane is C_k, covered $N_k = 2^{1-k}$ times.
- The height s remembers the passage: $\beta = \left\lfloor N_k \frac{s}{4\pi\rho_0} \right\rfloor + 1$.
- As $k \to -\infty$ the coil has 2^{n+1} turns of radius $\rho_0/2^n$: it collapses onto the vertical segment $\{O\} \times [0, 4\pi\rho_0)$.



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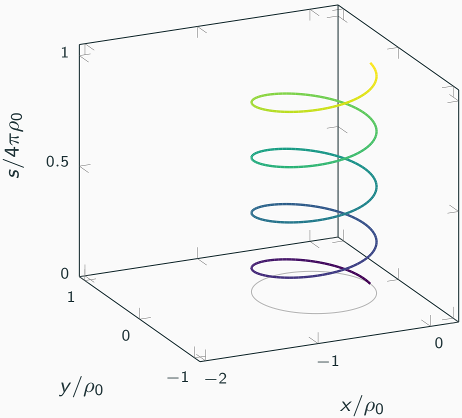
$k = -1$: $N = 4$ turns



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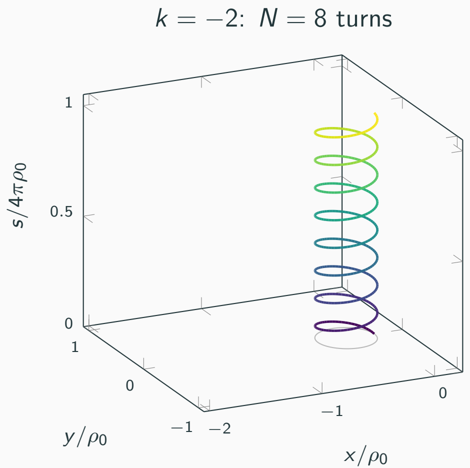

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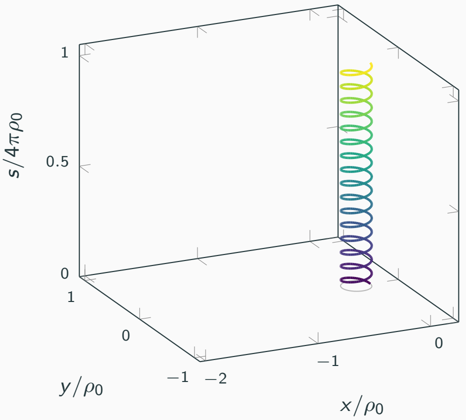
$k = -3$: $N = 16$ turns



A loop covered 2^{n_∞} times: lifting the covering index

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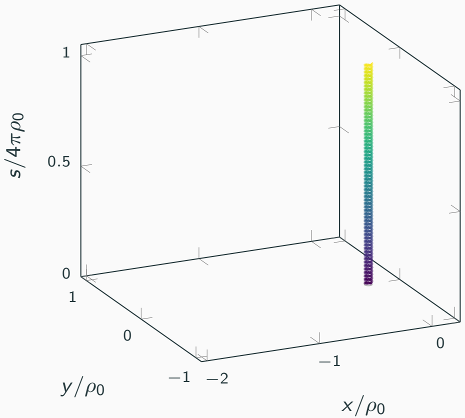
- As $k \to -\infty$ the coil has 2^{n+1} turns of radius $\rho_0/2^n$: it collapses onto the vertical segment $\{O\} \times [0, 4\pi\rho_0)$.

$k = -5$: $N = 64$ turns



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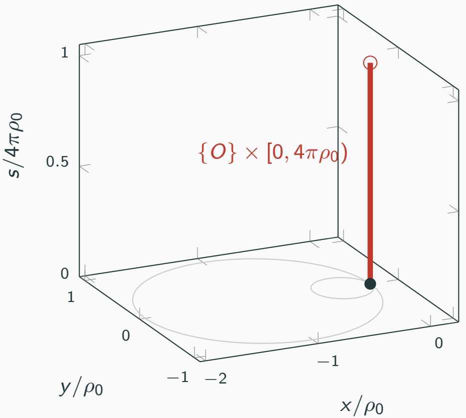
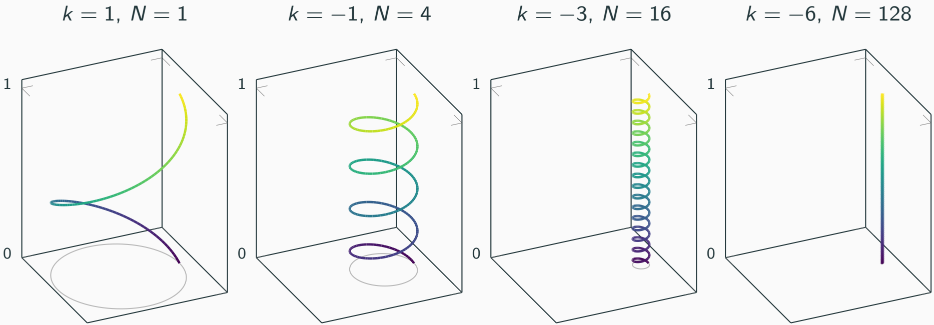
$k \to -\infty$: $N = 2^{n_\infty}$ turns

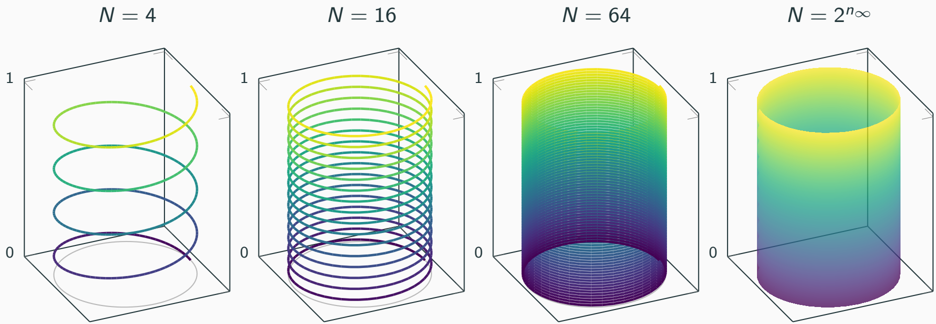


Side by side: the coil tightens



Same axes, $x, y \in [-2\rho_0, 0] \times [-\rho_0, \rho_0]$ (here the $k = 1$ circle is drawn at scale $\frac{1}{2}$ to fit). Colour $=$ height s, i.e. the covering index. The total length stays $4\pi\rho_0$ while the horizontal extent $2\rho_k \to 0$.

The same loops seen at their own scale



- Rescale by $1/\rho_k$ and normalise height $t = s/4\pi\rho_0 \in [0, 1)$: a unit circle covered N_k times, pitch $1/N_k$.
- As $N_k = 2^{n+1} \to 2^{n_\infty}$ the turns fill a cylinder $S^1 \times [0, 1]$: the infinitely covered loop, seen from inside.

A proposed reading of the open question

- The paper asks for a limit object at $k \to -\infty$ that "remembers" the covering index. The lift $\Gamma_k = (\gamma_k, s)$ is one natural candidate:

$$\Gamma_k \longrightarrow \big(O,\, s\big) \text{ uniformly}, \qquad \text{limit} = \{O\} \times [0, 4\pi\rho_0).$$

- s is ψ-invariant: ψ doubles the angle and halves the radius, so it preserves arc length ($\rho_{k-1} \cdot 2\theta = \rho_k \theta$).
- The index is a binary address: $\beta_k - 1 = \lfloor 2^{1-k} t \rfloor$ gives the first $1-k$ binary digits of $t = s/4\pi\rho_0$. A loop covered 2^{n_∞} times has as "index" an infinite binary word, i.e. a real $t \in [0, 1)$.

Two limits compared

	$k \to +\infty$	$k \to -\infty$
radius ρ_k	$\to \infty$	$\to 0$
turns $N_k = 2^{1-k}$	$\to 0$	$\to \infty$
limit in the plane	segment $[O; P)$, length $4\pi\rho_0$	point O
limit of the lift (γ_k, s)	the same segment	segment $\{O\} \times [0, 4\pi\rho_0)$

The lift (γ_k, s) and its limit are a suggestion added for this talk; they are not in the paper.

Conclusion

- A single family $(\mathcal{B}_k)$ of curves of equal length $4\pi\rho_0$: arcs, circle, multi-loops.
- The map ψ — radial factor $\cos\theta$, i.e. projection of O, i.e. angle doubling — links them: $\psi(\mathcal{B}_k) = \mathcal{B}_{k-1}$.
- The variant ψ^* stretches a half-open half-circle onto a whole circle, and clarifies three coexisting views of the circle: open, half-open, closed.
- Viète's formula appears as a metric corollary: the end points of the arcs climb to $4\pi\rho_0$.
- At $k \to -\infty$, the loop covered 2^{n_∞} times suggests a limit object with infinite covering — a segment "above" the point O.

References

[1] F. Viète, *Variorum de rebus mathematicis responsorum, liber VIII*, Tours, 1593.

[2] Euclid, *Elements*, Book I, definition 15.

[3] H. Genvrin, *Une famille de boucles circulaires emboîtées : homéomorphismes, formule de Viète et statut topologique du cercle*, 2004.