Proper Part and Partial Part

Finite, unbounded, infinite in the integers

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Outline

1. The whole and the part
2. The framework: a nonstandard model of Peano
3. Partial part, proper part
4. Finite, unbounded, infinite
5. Level of observation and resolution
6. The case of the rationals

Thesis

The same part of $\mathbb{N}$ is **unbounded without being infinite** as a partial part, and **infinite** as a proper part. It is not the set that changes, but the *grain* under which it is viewed.

The whole and the part

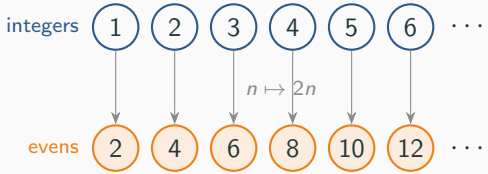
Cantor's choice

The bijection $n \mapsto 2n$ maps the integers onto the even numbers.

- For Cantor: same cardinality.
- The arithmetic of infinite cardinals is idempotent:

$$\aleph_0 + \aleph_0 = \aleph_0, \qquad 2 \cdot \aleph_0 = \aleph_0.$$

- The price: giving up Euclid's fifth common notion.



Euclid: the whole is greater than the part

	Cantor	This work
Size of the parts of $\mathbb{N}$	cardinality	numerosity
Arithmetic	idempotent: $\aleph_0 + \aleph_0 = \aleph_0$	non-idempotent: $\nu + \nu \neq \nu$
Evens and integers	same size	$\#\{\text{evens}\} < \#\mathbb{N}$
Euclid's common notion	given up	restored
Framework	set theory	Peano arithmetic

What is at stake

The two viewpoints do not contradict each other: they do not measure the same thing. Cardinality ignores the *grain*; numerosity counts at the grain of individuation of the integers.

Reminder: the definition of the infinite

Definition 1 (Infinite, basic definition)

A set E is *infinite* if the set of its parts $\mathcal{P}(E)$ can index all the points of a continuum.

- For Cantor, *every* infinite part of $\mathbb{N}$ satisfies this definition: $\mathcal{P}(F) \simeq [0,1]$ as soon as F is infinite in the usual sense.
- Question: what becomes of it once the grain of observation is fixed?

Announced answer

At the grain of the integers, $\mathbb{N}$ is the **smallest** infinite set.

Warning: two views of the same part

In common usage, a *proper part* (proper subset) is a part strictly included in the set. Here the expression is used in the sense of grain theory.

Partial part

$F \subset E$ regarded as **included** in E, at the base grain of E.

Proper part

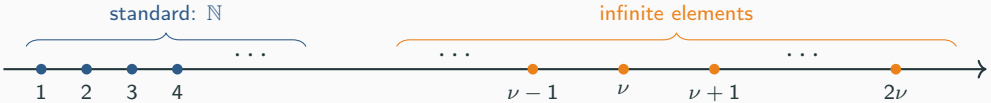
F regarded as **a whole**, renumbered by a bijection with E.

This is not a property of the set, but of the way it is viewed.

The framework: a nonstandard model of Peano

Peano integers and Skolem's theorem

- $\mathbb{N} = \{1, 2, 3, \ldots\}$, with addition, multiplication, order, and the first-order induction schema.
- **Skolem (1934)**: there are models $\mathcal{M}$ that satisfy *exactly* the same arithmetical statements as $\mathbb{N}$, and that contain elements larger than all the usual integers.
- $\mathbb{N}$ is an initial segment of $\mathcal{M}$: *standard* elements, then *infinite* elements.



sketch: no bound separates the two regions, there is no smallest infinite integer

Transfer and non-idempotence

Lemma 2 (Transfer)

Every arithmetical statement true in $\mathbb{N}$ is true in $\mathcal{M}$, and conversely, including those that name standard integers ($1+1+\cdots+1$).

This is the very definition of $\mathcal{M}$. The lemma plays the role of Robinson's transfer principle, but only concerns arithmetic.

Lemma 3 (Non-idempotence)

In $\mathcal{M}$, for every $x \neq 0$: $\quad x + x \neq x \quad$ *and* $\quad x - 1 < x$.

Proof.

A theorem of Peano: $x + x = x$ gives $x = 0$ by cancellation, and $x = (x-1)+1 > x-1$. True in $\mathbb{N}$, hence in $\mathcal{M}$. □

The integer $\nu = \#\mathbb{N}$

We fix an infinite integer $\nu \in \mathcal{M}$: it will be the numerosity of $\mathbb{N}$.

What we use

- the **order**: $\nu - 1 < \nu < \nu + 1 < 2\nu$;
- the **changes of level**: $2^\nu \cdot 2^\nu = 2^{2\nu}$.

What we do not use

The divisibility or the parity of ν: they would treat 2^ν as a completed integer.

We start from the given continuum, not from a dyadic construction converging to it. Values such as $\nu/2$ or $\sqrt{\nu}$ are understood up to one unit (integer parts): only their orders of magnitude matter.

Definable parts

Definition 4

A *definable* part of $\mathbb{N}$ is $F = \{n \in \mathbb{N} : \varphi(n)\}$, where φ is a formula of arithmetic. We write ${}^*F = \{x \in \mathcal{M} : \mathcal{M} \models \varphi(x)\}$.

Part	Formula $\varphi(n)$
evens	$\exists k \;\; n = 2k$
squares	$\exists k \;\; n = k \cdot k$
primes	$n > 1 \;\wedge\; \forall a \forall b \; (n = ab \Rightarrow a = 1 \vee b = 1)$

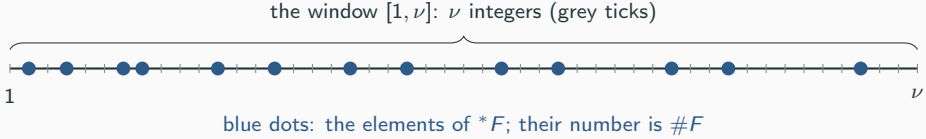
By transfer, ${}^*F \cap \mathbb{N} = F$: the same formula, read in $\mathcal{M}$, extends F beyond the standard.

Throughout, the parts under consideration are definable.

Definition 5 (Numerosity, mean grain)

The *window* is the interval $[1, \nu]$ of $\mathcal{M}$.

$$\#F = \left|{}^*F \cap [1, \nu]\right|, \qquad \text{mean grain of } F = \frac{\nu}{\#F}.$$



- The counting takes place *inside* $\mathcal{M}$: "number of elements of F at most n" is definable, evaluated at $n = \nu$.
- The mean grain is the thickness to give each element to tile the window: $L = |L| \cdot \varepsilon$.

The parts of a part of the window

A part of $[1, n]$ is coded by a number $< 2^n$: its binary expansion.



sketch: the k-th digit is 1 if and only if $k \in A$

Convention

The parts of a set of m elements of the window are represented by 2^m numbers of $\mathcal{M}$.
"The set of parts" is understood in this sense: it stays within arithmetic.

Properties of numerosity

(N1) $\#\mathbb{N} = \nu$;

(N2) if F is finite, $\#F$ is its usual number of elements;

(N3) if $F \cap G = \emptyset$, $\#(F \cup G) = \#F + \#G$;

(N4) if $F \subsetneq \mathbb{N}$, then $\#F < \nu$.

Proof of (N4).

There is a standard $m \notin F$: $\varphi(m)$ is false in $\mathbb{N}$, hence in $\mathcal{M}$. Since $m \leq \nu$, m is missing from ${}^*F \cap [1, \nu]$:

$$\#F \leq \nu - 1 < \nu.$$

□

(N4) is Euclid's common notion

The whole is greater than the part, even when only one element is missing:
$\#(\mathbb{N} \setminus \{1\}) = \nu - 1 < \nu, \quad \#\{\text{evens}\} \approx \nu/2 < \nu.$

Partial part, proper part

The even numbers as a partial part

The evens are part of a whole, the integers.

- In the window $[1, \nu]$: $\nu/2$ evens.
- Mean grain: 2.
- Each even number occupies one base grain.



blue disks: the evens among the integers, at the base grain

The even numbers as a proper part

Renumbered by $n \mapsto 2n$, the evens form a whole.

- The first ν evens occupy the window $[1, 2\nu]$.
- Grain 2: each red grain encloses two base grains.

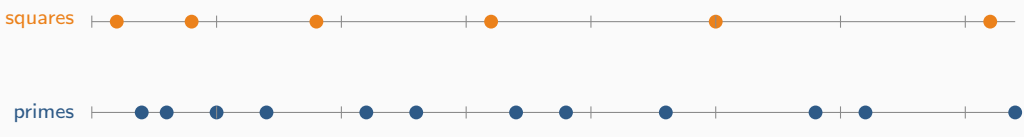


red circles: the evens renumbered, as a whole

A homogeneous level of observation

The regime changes with a level distinct from the grain of the integers, but **homogeneous**: all red grains have the same thickness, twice the base grain.

Squares and primes: heterogeneous grains



sketch: the gaps between consecutive elements vary

Squares

$\lfloor\sqrt{\nu}\rfloor$ squares in the window.
Mean grain $\simeq \sqrt{\nu}$: infinitely large.

Primes

${}^*\pi(\nu)$ primes; by transfer of the prime number theorem, ${}^*\pi(\nu)\ln\nu/\nu \simeq 1$.
Mean grain $\simeq \ln\nu$: infinitely large.

They too can be viewed as partial parts or as proper parts, but their grain is not homogeneous.

The general case

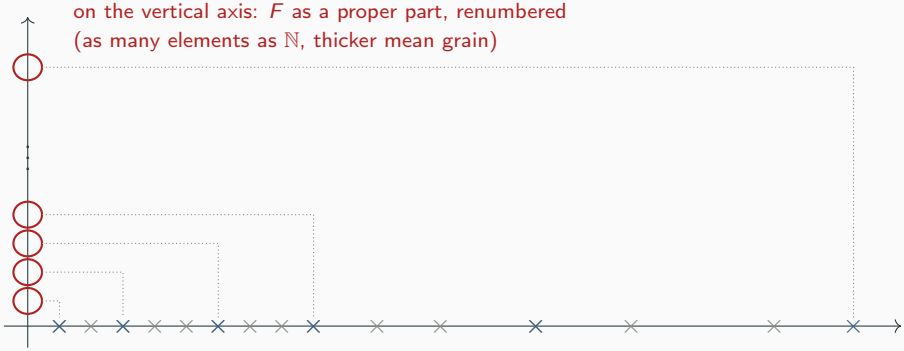
Let F be definable and infinite in the usual sense, $f : \mathbb{N} \to F$ its increasing enumeration ($f(n)$ is the n-th element), ${}^*f$ its extension to $\mathcal{M}$.

Definition 6 (The two views)

- F **as a partial part**: ${}^*F \cap [1, \nu]$, of numerosity $\#F$.
- F **as a proper part**: ${}^*f([1, \nu])$, its first ν elements, renumbered by f. It has ν elements and occupies the window $[1, {}^*f(\nu)]$.

	Partial part	Proper part
Window	$[1, \nu]$ (fixed)	$[1, {}^*f(\nu)]$ (enlarged)
Number of elements	$\#F \leq \nu$	ν
Mean grain	$\nu/\#F$	${}^*f(\nu)/\nu$

From one view to the other, in a picture



Sketch. As a proper part, F is indexed by as many elements as $\mathbb{N}$, at the cost of a thicker mean level of observation.

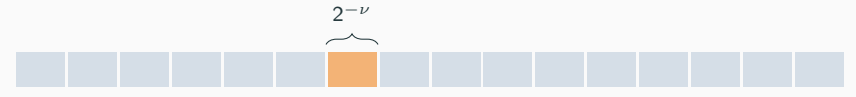
Finite, unbounded, infinite

The base continuum

For the definition of the infinite to distinguish the two views, the reference continuum is fixed at the grain of the integers.

Definition 7 (Base continuum)

The segment $[0, 1]$ taken at resolution $-\nu$: the 2^ν grains $\left[k2^{-\nu}, (k+1)2^{-\nu}\right]$, $0 \leq k < 2^\nu$, of thickness $2^{-\nu}$. This is the grid $\mathbb{G}_\nu$.



sketch: the 2^ν grains are represented by a few

The parts of the window $[1, \nu]$ are coded by 2^ν numbers: they index these grains *bijectively* (ν binary digits after the point).

Finite, unbounded, infinite

Definition 8 (Trichotomy)

A definable part $F \subset \mathbb{N}$, regarded as a partial part, is:

- **finite** if $\#F$ is a standard integer;
- **unbounded** if it is not finite;
- **infinite** if it is unbounded *and* its parts index the grains of the base continuum:

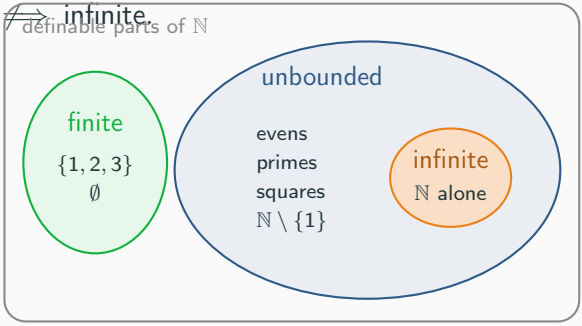
$$2^{\#F} \;\geq\; 2^{\nu}.$$

The condition $2^{\#F} \geq 2^{\nu}$ expresses that there is a surjection from the parts of ${}^{*}F \cap [1, \nu]$ onto the grains of the base continuum.

An inclusion, not a partition

infinite $\Longrightarrow$ unbounded, unbounded $\not\Longrightarrow$ infinite.

- Unboundedness is the **general** property of non-finite parts.
- The infinite is a **special case** of it.
- Three disjoint regimes: finite, unbounded non-infinite, infinite.



Main proposition

Proposition 9

Let $F \subset \mathbb{N}$ be definable, regarded as a partial part.

(i) *F is infinite if and only if $F = \mathbb{N}$;*

(ii) *F is finite if and only if F is finite in the usual sense;*

(iii) *every part of $\mathbb{N}$ that is infinite in the usual sense and different from $\mathbb{N}$ is unbounded without being infinite.*

Corollary

$\mathbb{N}$ is the **smallest** infinite set.

Proof

(i) $\#\mathbb{N} = \nu$: the parts of the window are coded by exactly 2^ν numbers, and index the grains of the base continuum bijectively.

If $F \subsetneq \mathbb{N}$, (N4) gives $\#F < \nu$, hence $2^{\#F} < 2^\nu$: too few parts, even for a surjection.

(ii) If F is finite, with largest element k: "every element of F is $\leq k$" is true in $\mathbb{N}$, hence in $\mathcal{M}$. Thus ${}^*F = F$ and $\#F$ is the usual number.

If F is infinite in the usual sense, it has at least n elements $\leq \nu$ for every standard n: $\#F$ is not standard.

(iii) follows from (i) and (ii).

The proof uses only the **order**: $\#F < \nu \Rightarrow 2^{\#F} < 2^\nu$.

The table of partial parts

Part	Numerosity	Mean grain	Parts	Status
integers	ν	1	2^{ν}	infinite (hence unbounded)
$\mathbb{N} \setminus \{1\}$	$\nu - 1$	$\simeq 1$	$2^{\nu-1}$	unbounded non-infinite
evens	$\nu/2$	2	$2^{\nu/2}$	unbounded non-infinite
primes	$\simeq \nu / \ln \nu$	$\simeq \ln \nu$	$2^{^*\pi(\nu)}$	unbounded non-infinite
squares	$\lfloor \sqrt{\nu} \rfloor$	$\simeq \sqrt{\nu}$	$2^{\lfloor \sqrt{\nu} \rfloor}$	unbounded non-infinite

$a \simeq b$: the ratio a/b differs from 1 by less than $1/n$ for every standard n. The values $\nu/2$, $\sqrt{\nu}$, $\ln \nu$ are understood as their integer parts, up to one unit.

The proper part becomes infinite again

Proposition 10

*Every definable part of $\mathbb{N}$ that is infinite (in the usual sense), regarded as a **proper part**, is infinite.*

Proof.

As a proper part, F is ${}^*f([1,\nu])$. By transfer of the injectivity of f, ${}^*f$ is injective: F has ν elements. Its parts are coded by 2^ν numbers, which index the grains of the base continuum bijectively. □

The thesis

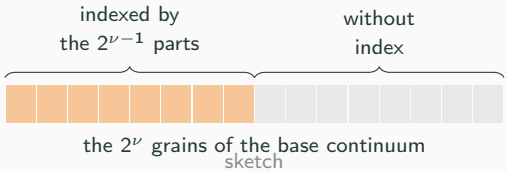
The same part of $\mathbb{N}$ is **unbounded without being infinite** as a partial part, and **infinite** as a proper part. It is not the set that changes, but the grain under which it is viewed.

A strict definition

$\mathbb{N} \setminus \{1\}$ has numerosity $\nu - 1$:

- it is unbounded;
- it is not infinite: $2^{\nu-1} < 2^{\nu}$.

Removing a single element is enough to lose infinity.



The counterpart of Euclid's principle

The part is strictly smaller than the whole, even when only one element is missing; and $2^{\nu-1}$ is only half of 2^{ν}.

Unbounded and infinite in Riemann

Riemann, habilitation lecture (Göttingen, 1854): one must distinguish the unboundedness (Unbegrenztheit) of space from its infinitude (Unendlichkeit).

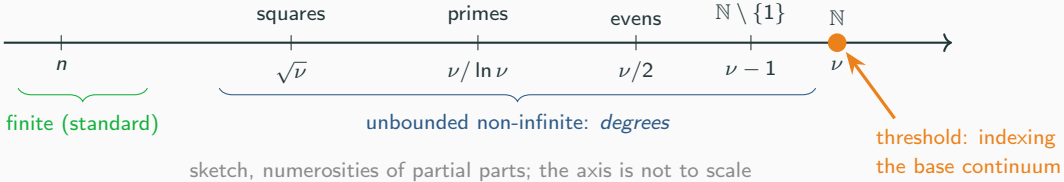
In Riemann

A sphere is unbounded (without boundary) without being infinite (its extent is finite). The distinction concerns the extent of space.

Here

An unbounded non-infinite part has no bound, but its parts are **not enough** to resolve the continuum at the grain of the integers.

Unboundedness is graded, the infinite is a threshold



sketch, numerosities of partial parts; the axis is not to scale

The infinite: absolute

Indexing the base continuum: a part has it or does not.

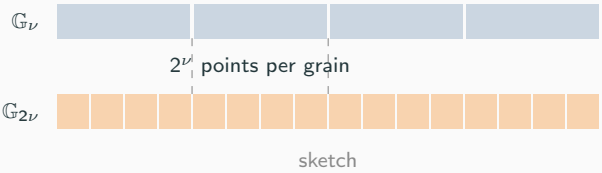
Unboundedness: graded

Measured by numerosity: primes $<$ evens $<$ $\mathbb{N} \setminus \{1\}$.

More unbounded without being more infinite

Beyond the threshold, unboundedness keeps growing; the infinite, for its part, is acquired.

- $\mathbb{G}_\nu$: 2^ν points.
- $\mathbb{G}_{2\nu}$: $2^{2\nu}$ points.
- $\mathbb{G}_{2\nu}$ is more unbounded than $\mathbb{G}_\nu$.
- Both index the continuum: they are **equally infinite**.



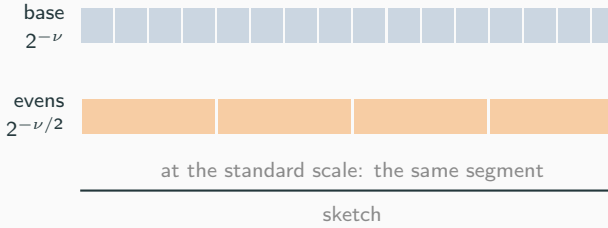
Self-similarity of ratio 2^ν extends the hierarchy step by step: $\mathbb{G}_\nu \subset \mathbb{G}_{2\nu} \subset \mathbb{G}_{3\nu} \subset \cdots$, each step adding ν digits.

Level of observation and resolution

The partial part resolves a coarser continuum

The $2^{\nu/2}$ parts of the evens index $[0, 1]$ at resolution $-\nu/2$: grains of thickness $2^{-\nu/2}$.

- $2^{-\nu/2}$ is still **infinitesimal**: $< 1/n$ for every standard n.
- Every point of the base continuum is infinitely close to a point of this coarser continuum.
- At the standard scale, the two continua cannot be told apart.



sketch

A status relative to the grain of the integers

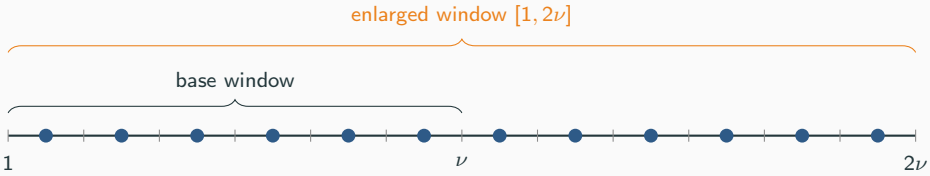
The status of unbounded non-infinite is not absolute: it is relative to the grain of the integers. An unbounded non-infinite part resolves the continuum, but with a thicker grain than the one fixed by $\mathbb{N}$. This is the relation $L = |L| \cdot \varepsilon$ applied to the window:

$$\nu = \frac{\nu}{2} \cdot 2 \quad \text{(evens)}, \qquad \nu \simeq \frac{\nu}{\ln \nu} \cdot \ln \nu \quad \text{(primes)}, \qquad \nu \simeq \sqrt{\nu} \cdot \sqrt{\nu} \quad \text{(squares)}.$$

Part	Parts	Resolution	Resolved grain
integers	2^{ν}	$-\nu$	$2^{-\nu}$ (base)
evens	$2^{\nu/2}$	$-\nu/2$	$2^{-\nu/2}$
squares	$2^{\lfloor\sqrt{\nu}\rfloor}$	$-\lfloor\sqrt{\nu}\rfloor$	$2^{-\lfloor\sqrt{\nu}\rfloor}$

The proper part and its window

As a proper part, the first ν evens occupy the window $[1, 2\nu]$.



sketch: grey ticks, the integers; blue dots, the evens

The integers of $[1, 2\nu]$

$2^{2\nu}$ parts: 2ν binary digits after the point.

Finer resolution -2ν: the grid $\mathbb{G}_{2\nu}$.

The evens of $[1, 2\nu]$

2^{ν} parts: ν binary digits.

Base resolution $-\nu$: the grid $\mathbb{G}_{\nu}$.

Lexical order and geometric order

- Enlarging the window refines the resolution: a finer lexical order, that of 2ν digits, corresponds to a finer geometric order of the points.
- This refinement is carried by the **integers** of the enlarged window, not by the evens themselves.
- Relative to the length they occupy, the evens keep a double grain.

The link with the constructed grain

Going from $[1, \nu]$ to $[1, 2\nu]$ means going from $\mathbb{G}_\nu$ to $\mathbb{G}_{2\nu}$: each base grain then contains 2^ν points, all infinitely close to its lower bound. The proper part *sees* level ν; the window it occupies *contains* level 2ν.

The case of the rationals

Fractions in the window

A fraction p/q is represented by the pair $(p, q) \in \mathbb{N}^2$.

Definition 11 (Numerosity in $\mathbb{N}^2$)

For a definable part $F \subset \mathbb{N}^2$:

$$\#F = \left|{}^*F \cap [1, \nu]^2\right|, \qquad \text{mean grain} = \frac{\nu^2}{\#F}.$$

All fractions

$\#\mathbb{Q}_+^* = \nu^2$: each numerator carries ν denominators. Mean grain 1.

The irreducible ones

$\mathbb{Q}_+^{\mathrm{irr}} = \{(p, q) : p \wedge q = 1\}$, definable: the gcd is a function of arithmetic.

Counting by numerator

Lemma 12

For standard p, the number of $q \in [1, \nu]$ coprime to p is

$$\nu \frac{\varphi(p)}{p} \quad \textit{up to } p.$$

$\lfloor \nu/p \rfloor$ complete blocks of p integers, each containing $\varphi(p)$ integers coprime to p; only the last block escapes the count. True in $\mathbb{N}$, hence in $\mathcal{M}$. Neither divisibility nor parity of ν.

Examples

- $p = 1$: ν denominators.
- p prime: $\simeq \nu - \nu/p$.

Pitfall

This is not ${}^*\pi(\nu) \simeq \nu/\ln \nu$: being prime *to* p is not being prime.

The total count

Proposition 13

$$\#\mathbb{Q}_+^{\mathrm{irr}} = \sum_{d=1}^{\nu} \mu(d) \left\lfloor \frac{\nu}{d} \right\rfloor^2 \;\simeq\; \frac{6}{\pi^2}\,\nu^2.$$

- **In $\mathbb{N}$**: Möbius inversion, then $A(n)/n^2 \to 1/\zeta(2) = 6/\pi^2$.
- **No reals in $\mathcal{M}$**: $x \simeq c\,\nu^2$ means $a\nu^2 < x < b\nu^2$ for all standard rationals $a < c < b$.
- **Transfer**: for fixed a, b, "$\forall n \geq N_0,\ an^2 < A(n) < bn^2$" is arithmetical; apply it at $n = \nu$.

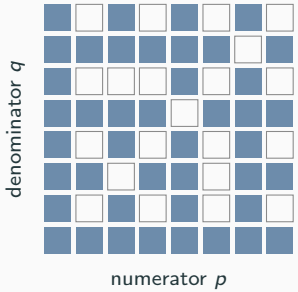
Check: $A(1000) = 608{,}383$, against $\frac{6}{\pi^2} \cdot 10^6 \approx 607{,}927$.

More irreducible than reducible fractions

$$\#\mathbb{Q}_+^{\mathrm{irr}} \simeq 0.608\,\nu^2, \qquad \#\mathbb{Q}_+^{\mathrm{red}} \simeq 0.392\,\nu^2$$

With $a = \frac{1}{2} < \frac{6}{\pi^2}$, an exact inequality in $\mathcal{M}$:

$$\#\mathbb{Q}_+^{\mathrm{red}} < \frac{\nu^2}{2} < \#\mathbb{Q}_+^{\mathrm{irr}}.$$



$[1,8]^2$: in blue, $p \wedge q = 1$ (43 out of 64)

Going from fractions to *values* loses only a **standard** proportion.

A standard grain

Part	Window	Numerosity	Mean grain	Grain
$\mathbb{N}$	$[1,\nu]$	ν	1	
evens	$[1,\nu]$	$\nu/2$	2	standard
primes	$[1,\nu]$	$\simeq \nu/\ln\nu$	$\simeq \ln\nu$	infinitely large
$\mathbb{Q}_+^*$	$[1,\nu]^2$	ν^2	1	
$\mathbb{Q}_+^{\mathrm{irr}}$	$[1,\nu]^2$	$\simeq \frac{6}{\pi^2}\nu^2$	$\simeq \pi^2/6 \approx 1.645$	standard

Like the evens

A **standard** mean grain.

Like the primes

A **heterogeneous** grain: 1 in column $p = 1$, $p/\varphi(p)$ in column p.

Two reference continua

	at the grain of $\mathbb{N}$ (resolution $-\nu$)	at the grain of $\mathbb{N}^2$ (resolution $-\nu^2$)
$\mathbb{Q}_+^*$	infinite	infinite, the smallest
$\mathbb{Q}_+^{\mathrm{irr}}$, $\mathbb{Q}_+^{\mathrm{red}}$ partial	infinite	unbounded non-infinite
$\mathbb{Q}_+^{\mathrm{irr}}$, $\mathbb{Q}_+^{\mathrm{red}}$ proper	infinite	infinite

At the grain of $\mathbb{N}^2$

The thesis holds again: unbounded as a partial part, infinite as a proper part.

At the grain of $\mathbb{N}$

The threshold becomes very low: the single column $p = 1$, with ν elements, is enough to index the base continuum.

In $\mathbb{N}$, removing one element loses infinity; in $\mathbb{N}^2$ at the grain of $\mathbb{N}$, almost everything can be removed. The status depends on the **reference continuum**.

The rationals, dense at the standard scale

Lemma 14 (Farey)

Let $\mathcal{F}_n$ be the sequence of irreducible $p/q \in [0,1]$ with $q \le n$ ($n \ge 2$). If $a/b < c/d$ are consecutive in it, then $bc - ad = 1$, $b + d > n$, and $\dfrac{1}{n(n-1)} \le \dfrac{c}{d} - \dfrac{a}{b} \le \dfrac{1}{n}$.



sketch: $\mathcal{F}_7$, the smaller the denominator, the taller the tick

Density

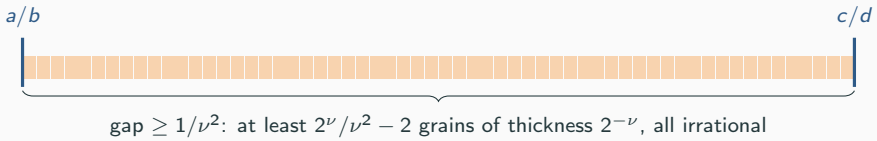
Every interval of length $\ge 1/\nu$ contains a rational of the window: the density of $\mathbb{Q}$ in $\mathbb{R}$ is recovered.

The rationals, rare at the grain of the continuum

A grain of the base continuum is rational if it contains a rational of the window, irrational otherwise.

Proposition 15 (Rarity)

- *At most* $2\,|{}^*\mathcal{F}_\nu| \leq 2\nu^2$ *rational grains, out of* 2^ν*: an* ***infinitesimal*** *proportion.*
- *Between two consecutive rationals, at least* $2^\nu/\nu^2 - 2$ *irrational grains: an* ***unbounded*** *number.*



What decides: ν^2 against 2^ν. The constant $3/\pi^2$ plays no role.

A density relative to the grain

Resolution	Grain thickness	The rationals of the window
standard	standard	dense
$-\lfloor \log_2 \nu \rfloor$	$\geq 1/\nu$	in every grain
$-\nu$	$2^{-\nu}$	in an infinitesimal proportion of the grains

Measure zero

Total length of the rational grains $\leq 2\nu^2 \cdot 2^{-\nu}$: standard part 0. This is "$\mathbb{Q}$ dense but of measure zero", with the order of magnitude.

Rationals at the grain of the window

In $\mathcal{M}$, every $k/2^\nu$ is a fraction. Only denominators $q \leq \nu$ are retained, as for numerosity.

Density, like the infinite, is not a property of the set, but of the grain.

Conclusion

Conclusion

1. **Euclid restored.** In nonstandard Peano, every strict part of $\mathbb{N}$ has a numerosity strictly smaller than $\nu = \#\mathbb{N}$.
2. **Three regimes.** Finite, unbounded non-infinite, infinite; the infinite is a special case of the unbounded.
3. **A minimum.** $\mathbb{N}$ is the smallest infinite set: evens, primes, squares, even $\mathbb{N} \setminus \{1\}$, are unbounded but not infinite as partial parts.
4. **Two views.** Renumbered as wholes, they become infinite again.
5. **Graded versus threshold.** Unboundedness is measured by numerosity; the infinite is the threshold for indexing the continuum.
6. **The rationals.** $\#\mathbb{Q}_+^{\mathrm{irr}} \simeq \frac{6}{\pi^2}\nu^2$, a standard grain; dense at the standard scale, rare at the grain of the continuum.

From partial part to proper part: not a change of set, but a **change of grain**.

Thank you

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