Pseudo-motion

From inertial integration to least action, via curvature

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The Grain Program

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Outline

Principle (Maupertuis's least action (Bruyset))

"Whenever a change occurs in nature, the quantity of action necessary for that change is the smallest that is possible."

Maupertuis, Œuvres, Jean-Marie Bruyset, Lyon, 1756, volume four, p. 36

Figure – Pierre Louis Moreau de Maupertuis (1698–1759).

Definition (Lagrangian)

The Lagrangian is the expression $\mathcal{L} = T - V$.

Definition (Lagrange's equations)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0.$$

Figure – Joseph-Louis Lagrange (1736–1813).

Definition (I.M. Gelfand, S.V. Formin “Calculus of variations“)

$$\frac{d}{dx}F_{y'} - F'_y = 0$$

Moreover, in what follows we bring out a construction partly analogous to Euler's, concerning the integration of a differential equation.

Figure – Leonhard Euler (1707–1783).

Hamilton - XIX°

Principle (Hamilton's least action)

$$S = \int_{t_1}^{t_2} \mathcal{L} \cdot dt \quad \text{such that } \delta S = 0.$$

Figure – William Rowan Hamilton (1805–1865).

The idea: concatenating segments of inertia

A dynamics $y = f(t)$ connects $O = f(t)$ to $A = f(t + n\Delta t)$. We build a *pseudo-motion* recursively:

1. at the point $f(t + (k-1)\Delta t)$, follow the **inertia vector** I_{k-1} carried by the *tangent* (the direction the dynamics would continue in *without force*);

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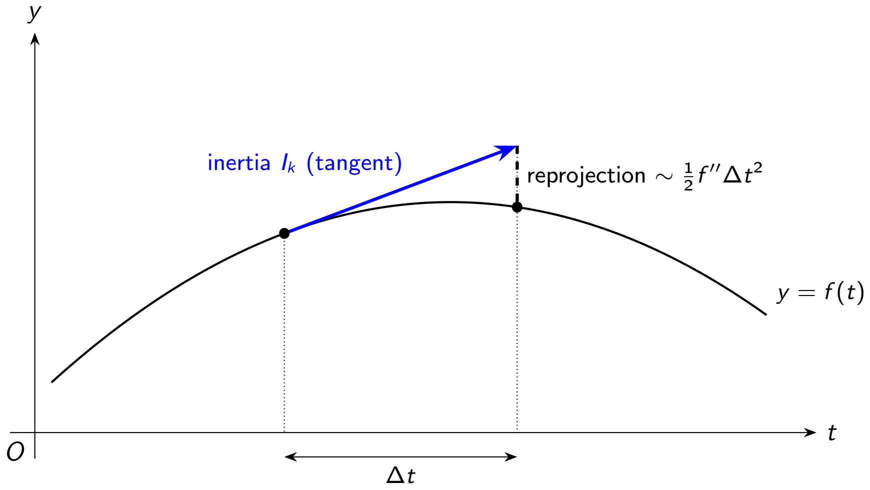
Taking the limit

As $\Delta t \to 0$, the concatenation of the local pseudo-motions *coincides* with the true dynamics.

The construction, in a picture



Blue: the inertia segment (free). Dashed: the reprojection (the force).

Proposition

The pseudo-motion is **Euler's method** for integrating the dynamics: advance along the derivative (tangent), then correct. As $\Delta t \to 0$, it converges to the real trajectory.

This is not *yet* least action

Euler convergence holds for *any* differential equation — including *dissipative* ones. It therefore cannot, on its own, imply that a trajectory minimizes an action. Two ingredients are missing:

- a **Lagrangian** $\mathcal{L} = T - V$ (hence a potential);
- a **comparison** between trajectories ($\delta S = 0$).

We will show that the construction *contains* them — under one condition.

The tangent/curve gap

$$f(t+\Delta t) - \underbrace{\big[f(t) + f'(t)\Delta t\big]}_{\text{tangent}} = \tfrac{1}{2}\, f''(t)\,\Delta t^2 + o(\Delta t^2).$$

The **reprojection** is governed by the *curvature* $f'' = \ddot{y}$. By Newton, $m\ddot{y} = F$: the reprojection is the **impulse of the force**.

The inertial intuition

On the *segment* alone: $f'' = 0$, zero gap — this is a **free least-action motion** ($V = 0$). "No force $\Rightarrow$ least action" is *correct for the segment*. But the global trajectory *alternates* inertia (free) and reprojection (force): the force is hidden in the reprojections. It is there, and only there, that it acts.

The bridge theorem

If the force is *conservative* ($F = F(y)$, position only)

1. it derives from a potential: $F = -\dfrac{dV}{dy} \Rightarrow V(y) = -\displaystyle\int F\,dy$ (the integrated curvature gives V);

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3. Euler–Lagrange $\dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{y}} - \dfrac{\partial \mathcal{L}}{\partial y} = 0$ recovers $m\ddot{y} = -\dfrac{dV}{dy} = F$: **Newton**, i.e. the construction itself.

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Conclusion

The trajectory makes the action $S = \int \mathcal{L}\,dt$ **stationary**: it satisfies the principle of **least action**. *The heuristic was right:* curvature $\rightarrow$ force $\rightarrow$ potential $\rightarrow$ Lagrangian $\rightarrow$ least action.

The boundary: conservativeness

The condition that separates true from false

$$\text{force } F = F(y) \text{ (position only)} \implies \text{potential, Lagrangian, least action.}$$

$$\text{force } F = F(\dot{y}) \text{ (velocity)} \implies \text{no } V(y), \text{ no standard Lagrangian.}$$

Practical test. Does the force read off the curvature depend on:

- only *where* we are (y)? $\Rightarrow$ potential, least action;
- also *how fast* ($\dot{y}$)? $\Rightarrow$ neither potential nor least action.

Friction $F = -k\dot{y}$: the construction remains a valid Newton integration, but least action does not apply.

Conservative case: parabolic fall

$$y(t) = v_0 t - \tfrac{1}{2} g t^2, \qquad m = 1,\ g = 9.81.$$

Step	Numerical result
Measured curvature $\ddot{y}$	-9.80 (expected $-g = -9.81$)
Force $F = m\ddot{y}$	-9.80 N (expected $-mg$)
Depends on $\dot{y}$?	no $\Rightarrow$ conservative
Potential $V(y) = -\int F\,dy$	mgy (gravity)
Euler–Lagrange $m\ddot{y} + mg$	$0.007 \approx 0$ verified

The chain curvature $\rightarrow$ force $\rightarrow$ potential $\rightarrow$ Euler–Lagrange is confirmed to numerical precision.

The action is stationary (it is a minimum)

$$S = \int \left(\tfrac{1}{2} m\dot{y}^2 - mgy \right) dt, \quad \text{perturbations } \varepsilon \sin(\pi t/T) \text{ (zero at endpoints).}$$

ε	-0.10	-0.05	**true**	$+0.05$	$+0.10$
S	-35.758	-35.767	-35.770	-35.767	-35.758
ΔS	$+0.0123$	$+0.0031$	0	$+0.0031$	$+0.0123$

Reading

All neighbours have *larger* S, and ΔS is symmetric, $\propto \varepsilon^2$: the signature of a **minimum**. The true trajectory makes the action stationary.

The code

```
import numpy as np
m, g = 1.0, 9.81
t = np.linspace(0, 2, 2001); y = 10*t - 0.5*g*t**2   # parabola
ydot = np.gradient(y, t); ydd = np.gradient(ydot, t) # velocity, curvature
F = m*ydd             # force = m * curvature
V = m*g*y             # V = -int F dy = mgy
EL = m*ydd + m*g      # Euler-Lagrange : ~0
trapz = getattr(np, 'trapezoid', np.trapz)
S = lambda yy: trapz(0.5*m*np.gradient(yy,t)**2 - m*g*yy, t)
for e in (-0.1, -0.05, 0.05, 0.1):
    b = e*np.sin(np.pi*t/t[-1])   # zero at endpoints
    print(e, S(y+b)-S(y))         # all > 0  =>  minimum
```

Dissipative case $F = -mg - k\dot{y}$: $\mathrm{correlation}(F, \dot{y}) = -1.000 \Rightarrow$ non-conservative force, least action inapplicable.

Transport by the force: neither projection nor reprojection

The reprojection *followed* a given curve. We replace it with a **generative** construction: the incremental Newton law.

Generative construction

Setting $q = m\dot{y}$ the momentum,

$$\frac{\Delta q}{\Delta t} = F \iff \boxed{q(t + k\Delta t) = q(t + (k-1)\Delta t) + F \cdot \Delta t}$$

We start from q, add the **impulse** $F\Delta t$, and obtain the new q.

The fundamental difference

This is *neither* a projection *nor* a reprojection: it is a **transport by the force**. It does not assume the curve is known — it *produces* the curve from F. The force is the *rate of change of momentum* ($F = \Delta q/\Delta t$), not "the change of the force".

The impulse triangle

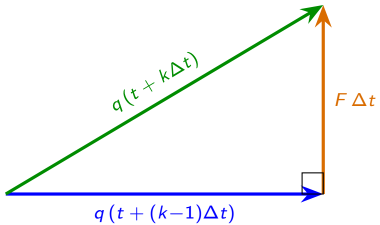
Geometric reading of the step

Each step is a vector triangle:

- side $q(t+(k-1)\Delta t)$ (inertia);
- side $F\Delta t$ (impulse);
- resultant $q(t+k\Delta t)$.

Convergence

For finite Δt, the endpoint is not on the curve (Euler error $\sim \Delta t^2$). As $\Delta t \to 0$, the triangle comes to **hug** the curve.



inertia + impulse = new momentum

The energy content: ΔT and ΔV

The momentum graph reveals the **energy**, where the position graph showed only the trajectory.

The two faces of the impulse

The work $F \cdot \Delta x$ is simultaneously:

$$\Delta T = F \cdot \Delta x \text{ (kinetic gained)}, \qquad \Delta V = -F \cdot \Delta x \text{ (potential released)},$$

hence the **step-by-step conservation of energy**:

$$\boxed{\Delta T + \Delta V = 0 \implies T + V = \text{const.}}$$

step n	ΔT	ΔV	$\Delta T + \Delta V$
100	-0.088	$+0.088$	$\sim 10^{-13}$
500	-0.050	$+0.050$	$\sim 10^{-14}$
1500	$+0.046$	-0.046	$\sim 10^{-12}$

Verified to machine precision

$\Delta T = -\Delta V$ at each step. The total energy $E = T + V$ is constant ($\Delta E/E \sim 10^{-3}$, discretization residual).

Position + momentum = phase space

Two complementary graphs

- **position** graph: the *trajectory*;
- **momentum** graph: the *energy exchange* (ΔT gained, ΔV released).

Together: the *phase space* (y, q), where Hamiltonian mechanics lives.

What the second graph carries

The Lagrangian $\mathcal{L} = T - V$ **and** its conservation $T + V =$ const. It is the *energetic counterpart* of the kinematic graph.

The position graph generates the shape;
the q graph generates the energy.

Consistent with the reading through the grain: inertia (T) + deformation (V).

Inertia + deformation

The dynamics *decomposes* into:

- **inertia** (segments) $\to$ kinetic T, free least action;
- **reprojection** (curvature) $\to$ force, potential V.

The reprojection is a *deformation* of the inertial trajectory, governed by f'' — the object of the *deformation gauge*.

The fold, again

Where the trajectory folds ($\dot{y} = 0$, extremum of y), the position gauge **loses fineness**: two instants, same height. This is the *fold* from the *Gauges* chapter.

Two languages, one trajectory: least action (Lagrangian) and deformation gauge (curvature).

What has been established

- The pseudo-motion is a **Newton integration** (Euler's method).
- Its **reprojection encodes the force**, through the curvature $f'' = \ddot{y}$.
- *If* the force is **conservative**: curvature $\to$ force $\to$ potential $\to$ Lagrangian $\mathcal{L} = T - V \to$ **least action**.
- Verified: Euler–Lagrange at 0.007, **stationary** action (minimum).
- The pseudo-motion reads as a **deformation gauge** of the dynamics.

Caveats

(i) Conservativeness required (force $= F(y)$); forces in $\dot{y}$ or t rule it out. *(ii)* The reprojection axis must be that of the force. *(iii)* A mechanical result: we link a geometric construction to Newton and Lagrange — the potential is *read off*, not predicted.

Thank you

The pseudo-motion: inertia + deformation

The Grain Program

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