Quantum Mechanics in the Grain Program

Hilbert space, Hopf sphere, the two postulates, and promotion

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September 10, 2026

Overview

The stratal Hilbert space $\widetilde{\mathbb{C}}^n$

Setting

States live in $\mathcal{H} = \widetilde{\mathbb{C}}^n$, a vector space over the stratal complex field $\widetilde{\mathbb{C}} = \widetilde{\mathbb{R}}[i]$, with a hermitian inner product $\langle\cdot|\cdot\rangle$.

- Normalized states form the *state sphere* $S^{2n-1} \subset \widetilde{\mathbb{C}}^n$.
- Each coefficient $c = |c|\, e^{i \arg c}$ splits stratally:
  - **modulus** $|c|$ on the real axis $\leftrightarrow \mathcal{G}_{-2n_\infty}$ (fine stratum);
  - **phase** $\arg c$ on the circle $\leftrightarrow \mathcal{G}_{-n_\infty}/\mathcal{G}_{-2n_\infty}$ (cyclic stratum).
- i is not adjoined algebraically: it *emerges geometrically* from the stratal duality (half-turn $\theta = \pi \leftrightarrow -1$).

The Hopf sphere: state sphere and phase fibre

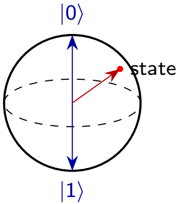
For a qubit ($n = 2$), the normalized states form $S^3 \subset \widetilde{\mathbb{C}}^2$. The **Hopf fibration**

$$S^1 \hookrightarrow S^3 \xrightarrow{\pi} S^2$$

separates two stratal roles:

- the **base** S^2 (Bloch sphere) = the physical state, up to global phase;
- the **fibre** S^1 = the global phase, carried by the cyclic stratum $\mathcal{G}_{-n_\infty}/\mathcal{G}_{-2n_\infty}$.

The phase that Hopf quotients out is exactly the stratal cyclic stratum.



Bloch sphere S^2

Postulate 1 — Stratal superposition

Postulate

The state space is $\mathcal{H} = \widetilde{\mathbb{C}}^{\,n}$, vectorial over $\widetilde{\mathbb{C}}$. If $|\phi_1\rangle, |\phi_2\rangle$ are states, so is

$$|\psi\rangle = \alpha|\phi_1\rangle + \beta|\phi_2\rangle, \quad \alpha, \beta \in \widetilde{\mathbb{C}},$$

after normalization $\||\psi\rangle\| = 1$.

A coefficient $\lambda = |\lambda| e^{i \arg \lambda}$ acts as **dilation–rotation**:

- dilation $|\lambda|$ on the fine stratum $\mathcal{G}_{-2n_\infty}$ (amplitude);
- rotation $\arg \lambda$ on the cyclic stratum $\mathcal{G}_{-n_\infty}/\mathcal{G}_{-2n_\infty}$ (phase).

Interference lives on the cyclic stratum

Norm of a superposition

$$\big\| |\psi\rangle \big\|^2 = |\alpha|^2 + |\beta|^2 + \underbrace{2\,\mathrm{Re}\big(\bar{\alpha}\beta\,\langle\phi_1|\phi_2\rangle\big)}_{I}$$

Writing $\bar{\alpha}\beta\langle\phi_1|\phi_2\rangle = R\,e^{i\Phi}$, one gets $I = 2R\cos\Phi$.

- Classical addition would give only $|\alpha|^2 + |\beta|^2$ (no cross term).
- I is the *content* of superposition — absent classically.
- Modulus R on $\mathcal{G}_{-2n_\infty}$; relative phase Φ on the **cyclic stratum** $\mathcal{G}_{-n_\infty}/\mathcal{G}_{-2n_\infty}$ — the same S^1 as the Hopf fibre.

Quantum interference and the Bloch phase share one stratum.

Postulate 2 — Stratal observables

Postulate

Every measurable quantity corresponds to a **hermitian operator** $A = A^\dagger$ on $\widetilde{\mathbb{C}}^n$. Its *eigenvalues* are the possible results; its *eigenvectors* form a decomposition basis of states.

Stratal spectral theorem

After promotion: eigenvalues λ_k are *real* ($\in \widetilde{\mathbb{R}}$); an orthonormal eigenbasis ($|a_k\rangle$) exists; $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

Real eigenvalues: $\lambda = \langle a|A|a\rangle = \overline{\langle a|A^\dagger|a\rangle} = \bar{\lambda}$. Hermiticity forces the spectrum onto the fine (real) stratum $\mathcal{G}_{-2n_\infty}$.

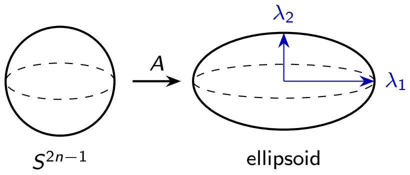
Geometric reading: sphere $\to$ ellipsoid

As a linear map, A sends the state sphere S^{2n-1} to an **ellipsoid** with semi-axes $|\lambda_k|$ along the eigendirections:

$$A(S^{2n-1}) = \left\{ \sum_k \lambda_k x_k |a_k\rangle : \sum_k |x_k|^2 = 1 \right\}.$$

- distortion is *real* (spectrum real);
- each axis dilated by λ_k (compressed if < 1, flipped if < 0).

Geometric reading of the spectrum — distinct from the physical effect of a measurement.



Stratal promotion and Born's rule

A state decomposes on the eigenbasis: $|\psi\rangle = \sum_k c_k |a_k\rangle$, $c_k = \langle a_k|\psi\rangle$.

Stratal localization

eigenvalues λ_k (results)	fine stratum $\mathcal{G}_{-2n_\infty}$ (real)
amplitudes $\lvert c_k \rvert$	fine stratum $\mathcal{G}_{-2n_\infty}$ (real)
phases $\arg c_k$	cyclic stratum $\mathcal{G}_{-n_\infty}/\mathcal{G}_{-2n_\infty}$

- **Born's rule** $|c_k|^2$: derived from the factor 2 of the logarithmic transform between adjacent strata.
- **Measurement** $=$ irreversible stratal projection $\mathcal{G}_{-2n_\infty} \to \mathcal{G}_{-n_\infty}$: collapse $=$ crushing of the phase circle. It is a *decomposition* (Landauer-irreversible), like the promotion st.

The correspondence principle, stratally

Promotion $\mathrm{st} : \widetilde{\mathbb{R}}_{\mathrm{fin}} \to \mathbb{R}$

Passing to the coarse observed reality erases the fine stratum (the phase, the infinitesimal fineness). This is the quantum $\to$ classical passage.

- Superposition and interference live on the *fine* + *cyclic* strata.
- Promotion crushes the cyclic stratum $\Rightarrow$ phases (hence interference) disappear $\Rightarrow$ classical probabilities remain.
- The **correspondence principle** is thus a *stratal projection*: the classical limit is the promoted (phase-erased) image of the quantum state.

$$\text{quantum (fine + cyclic)} \xrightarrow{\text{st}} \text{classical (coarse, phase-free)}$$

Summary

- **Hilbert** $\widetilde{\mathbb{C}}^n$: modulus on $\mathcal{G}_{-2n_\infty}$, phase on $\mathcal{G}_{-n_\infty}/\mathcal{G}_{-2n_\infty}$; i emerges geometrically.
- **Hopf** $S^1 \hookrightarrow S^3 \to S^2$: Bloch base $=$ state, fibre $=$ phase $=$ cyclic stratum.
- **Postulate 1**: superposition; interference $I = 2R\cos\Phi$ lives on the cyclic stratum.
- **Postulate 2**: hermitian observables; real spectrum; sphere $\to$ ellipsoid (semi-axes $|\lambda_k|$).
- **Promotion**: Born from factor 2; measurement $=$ irreversible stratal projection; correspondence principle $=$ phase-erasing promotion.

Status

An exploratory *stratal reading*: it does not derive quantum mechanics, but grounds its structures (complex states, Born, collapse, classical limit) in the grain program's stratification.

Thank you

Quantum Mechanics in the Grain Program