Quadratics in the Theory of the Grain

Proper part, partial part and the imaginary unit

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Outline

1. Origin of the complex plane
2. The roots of an irreducible quadratic
3. Taking Viète's formulas all the way
4. The underlying rotation group
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7. Imaginaries in the real world
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Guiding thread

The roots of a quadratic are read through two views: the **proper part**, measured in the reference unit, and the **partial part**, carried by an axis whose reference whole has changed. The imaginary unit is the *passage* from one view to the other: a quarter turn.

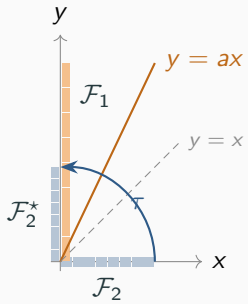
Origin of the complex plane

The reflexivity set-up

- $\mathcal{F}_2 = [0, 1[$ on the x-axis, 2^ν grains of thickness $\varepsilon = 2^{-\nu}$.
- τ: quarter turn, $\mathcal{F}_2^\star = \tau(\mathcal{F}_2)$, grains ε.
- $f : x \mapsto ax$: $\mathcal{F}_1 = [0, a[$, grains $a\varepsilon$.
- $h = f \circ \tau^{-1} : \mathcal{F}_2^\star \to \mathcal{F}_1$: the part–whole bijection.

Two views of $\mathcal{F}_2^\star$

partial: included in $\mathcal{F}_1$ at grain ε;
proper: renumbered as a whole by h.



From the real plane to the complex plane

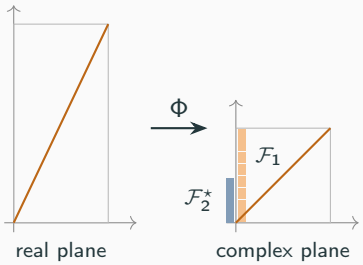
Real plane: a single norm

Both axes are measured with the unit $\mathcal{F}_2$. $\mathcal{F}_2^\star$ is seen there only as a *partial part* of $\mathcal{F}_1$.

Complex plane: changing the whole

Real axis: $\mathcal{F}_2$, proper part and unit. Vertical axis: $\mathcal{F}_1$ as proper part; $\mathcal{F}_2^\star$ becomes a partial part there.

Normalisation $\Phi(x, y) = (x,\ y/a)$:
$y = ax \ \mapsto \ y = x$.



The reflexivity operator

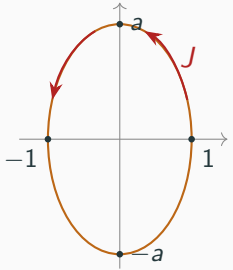
Definition 1

$J = h \circ \tau$ on the x-axis, $J = \tau \circ h^{-1}$ on the y-axis:

$$J(x, y) = \left(-\frac{y}{a},\ ax\right), \quad J = \begin{pmatrix} 0 & -1/a \\ a & 0 \end{pmatrix}.$$

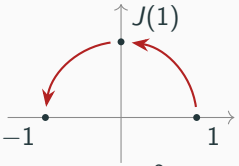
Theorem 2

$J^2 = \tau^2 = -I$: the dilation h cancels out. In the normalised plane, J is the quarter turn.



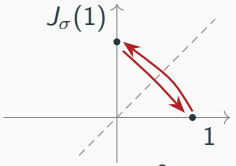
orbit: ellipse $x^2 + y^2/a^2 = 1$

Where the sign comes from: rotation or reflection



rotation τ: $J^2 = -I$

complex numbers



reflection σ: $J_\sigma^2 = +I$

hyperbolic numbers

The choice is not a convention

For counting reflexivity, rotation and reflection are equivalent. For roots they are not: with $j^2 = +1$, a quadratic with negative discriminant **has no root at all** ($\mathbb{R}[j] \simeq \mathbb{R} \times \mathbb{R}$).

The roots of an irreducible quadratic

Cayley–Hamilton and the companion matrix

Theorem 3 (Cayley–Hamilton)

Every matrix satisfies its characteristic polynomial. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$*:*

$$A^2 - (a+d)\,A + (ad - bc)\,I = 0.$$

Companion matrix of $x^2 + px + q$

$$C = \begin{pmatrix} 0 & -q \\ 1 & -p \end{pmatrix}, \quad \det(\lambda I - C) = \lambda^2 + p\lambda + q.$$

It means "multiplying by x" while replacing x^2 by $-px - q$:
$x \cdot 1 = x, \;\; x \cdot x = -q - px$.

For $x^2 + 1$, the rule $x^2 = -1$ turns x into the number i, and C is the rotation $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$: this is Cauchy's construction of the complex numbers (1847).

Proposition 4

If $\Delta = p^2 - 4q < 0$, set $\alpha = -\frac{p}{2}$, $\beta = \frac{\sqrt{|\Delta|}}{2}$. Then

$$C = \alpha I + \beta K, \qquad K^2 = -I,$$

K is conjugate to the rotation τ, C has no invariant line, and the roots are $\alpha \pm i\beta$.

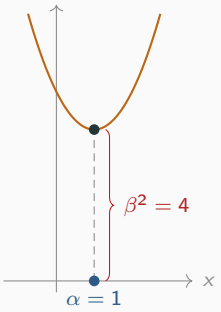
Proof. Cayley–Hamilton: $C^2 + pC + qI = 0$, hence $\left(C + \frac{p}{2} I\right)^2 = \frac{\Delta}{4} I = -\beta^2 I$.

Example: $x^2 - 2x + 5$

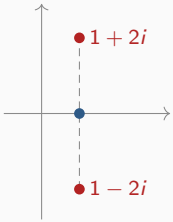
$$C = \begin{pmatrix} 0 & -5 \\ 1 & 2 \end{pmatrix}, \quad K = \tfrac{1}{2}(C - I) = \tfrac{1}{2}\begin{pmatrix} -1 & -5 \\ 1 & 1 \end{pmatrix}, \quad K^2 = -I, \quad C = I + 2K\text{: roots } 1 \pm 2i.$$

Reading the roots

- **Real part** α: the homothetic component; it stays on the real axis, in unit $\mathcal{F}_2$: proper part.
- **Imaginary part** β: the coefficient of K, which leads to the axis whose whole is $\mathcal{F}_1$: partial part.
- **Irreducible**: neither view carries the solution on its own.



real plane



complex plane

The real plane sees the abscissa α of the vertex and its height β^2: the square of the partial part, not its root.

Historical landmarks

Tool	Authors	Date
coefficient–root relations	Viète, Girard	1591–1629
imaginaries in the irreducible cubic case	Cardano, Bombelli	1545, 1572
complex plane	Wessel, Argand, Gauss	1799–1831
complex numbers as pairs of reals	Hamilton	1837
complex numbers as polynomials modulo $x^2 + 1$	Cauchy, then Kronecker	1847, 1882
Cayley–Hamilton theorem	Hamilton, Cayley, Frobenius	1853, 1858, 1878
normal forms, companion matrix	Weierstrass, Jordan, Frobenius	1868–1879

What is specific to the theory of the grain

The operator J built from h and τ, and the reading of the roots as proper part and partial part.

Taking Viète's formulas all the way

Viète's form

With $\Sigma = -\frac{b}{a}$, $\Pi = \frac{c}{a}$ and $\alpha = \frac{\Sigma}{2}$:

$$\frac{1}{a}P(x) = x^2 - \Sigma x + \Pi = (x-\alpha)^2 - (\alpha^2 - \Pi), \qquad x = \alpha \pm \sqrt{\alpha^2 - \Pi}.$$

The **margin** $D = \alpha^2 - \Pi$ decides everything:

- $\Pi < \alpha^2$: real roots $\alpha \pm \delta$;
- $\Pi = \alpha^2$: double root α;
- $\Pi > \alpha^2$: complex roots $\alpha \pm i\beta$.

Proper part: the mean α, in every case.

Quadratic	Π	Roots
$x^2 - 4x + 3$	3	1 and 3
$x^2 - 4x + 4$	4	2 (double)
$x^2 - 4x + 5$	5	$2 \pm i$

Discriminant: $b^2 - 4ac = 4a^2(\alpha^2 - \Pi)$.

The maximal product

Proposition 5

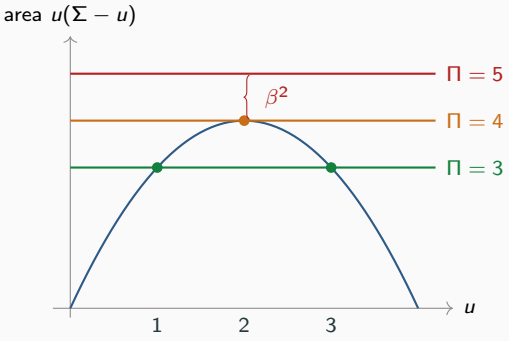
For $u + v = \Sigma$:

$$uv = \alpha^2 - \left(\frac{u - v}{2}\right)^2 \leq \alpha^2.$$

The square has the largest area among rectangles with the same half-perimeter.

Meaning of the imaginary

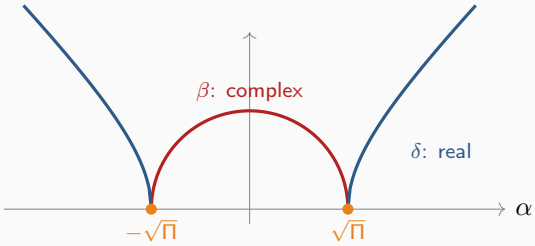
Complex roots $\iff$ the required product exceeds the real maximum.
Partial part: the excess, $\beta^2 = \Pi - \alpha^2$.



Hyperbolic or circular

- Real roots: $\Pi = \alpha^2 - \delta^2$, hyperbolic, like $(\alpha + j\delta)(\alpha - j\delta)$.
- Complex roots: $\Pi = \alpha^2 + \beta^2 = |\alpha + i\beta|^2$, circular; $\sqrt{\Pi}$ is a hypotenuse.

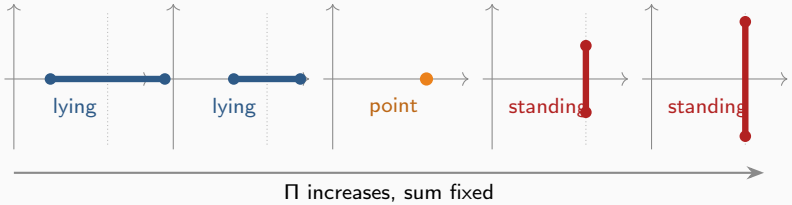
For a fixed product, the half-spread decreases along the hyperbola, vanishes at the double root, and grows again along the circle.



The grain of the roots

Definition 6

The *grain of the roots* is the segment joining the two roots; its centre is the proper part α, its half-spread $\sqrt{|\alpha^2 - \Pi|}$.



Lying when the roots are real, reduced to its centre for a double root, raised by a quarter turn when they are complex. Only the direction changes.

Conjugation: a symmetry of the partial parts

- Conjugation fixes the real axis and reverses the imaginary axis: it exchanges J and $J^{-1} = -J$, that is, τ and τ^{-1}.
- A real quadratic does not distinguish the two directions of the passage from part to whole: its roots come in pairs.

What the real plane sees (Viète)

$$(\alpha + i\beta) + (\alpha - i\beta) = 2\alpha, \qquad (\alpha + i\beta)(\alpha - i\beta) = \alpha^2 + \beta^2.$$

In the sum, the partial parts cancel; in the product, they appear only squared. The real plane knows neither β nor its sign.

The underlying rotation group

The plane driven by a rotational motion

- The quarter turn $i = \mathrm{Rot}(O, \pi/2)$ generates

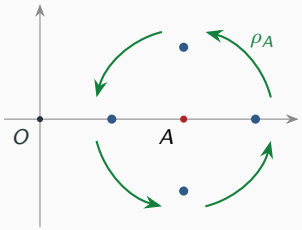
$$G = \langle i \rangle = \{\mathrm{Id}, i, i^2, i^3\} \simeq \mathbb{Z}/4\mathbb{Z},$$

  which takes the plane through four states.

- Fixed point O; barycentre of the orbit: $1 + i + i^2 + i^3 = 0$.

- The change of frame $(O, 1, i) \mapsto (A, 1, i)$, with $A = (\mu_a, 0)$, transports this motion:

$$\rho_A : z \mapsto \mu_a + i\,(z - \mu_a), \qquad G_A = \langle \rho_A \rangle \simeq G.$$



A quadratic never has both real and complex roots

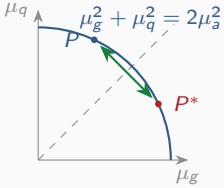
Centred form: with $X = x - \mu_a$, $P(X + \mu_a) = X^2 - D$, $D = \mu_a^2 - \mu_g^2$.

Definition 7

The **dual** of $P = x^2 - \Sigma x + \Pi$ is $P^* = x^2 - \Sigma x + (2\mu_a^2 - \Pi) = X^2 + D$.

Proposition

- Same centre, opposite margin; $(P^*)^* = P$.
- $P^* = x^2 - 2\mu_a x + \mu_q^2$: the dual exchanges μ_g and μ_q with μ_a fixed.
- If $D \neq 0$: one has real roots, the other complex.
- $P \cdot P^* = X^4 - D^2$.



The square of roots

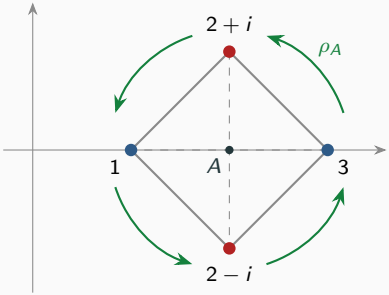
Theorem 8

For $D \neq 0$ and $d = |D|$, the roots of P and P^* are

$$\mu_a + i^k \sqrt{d}, \qquad k = 0, 1, 2, 3 :$$

the vertices of a square with centre A and side $\sqrt{2d}$: the orbit of $\mu_a + \sqrt{d}$ under G_A, with barycentre A.

Example: $x^2 - 4x + 3$ (roots $1, 3$) and its dual $x^2 - 4x + 5$ (roots $2 \pm i$).



The rotation group underlying a quadratic

Theorem 9

For every real quadratic with margin $D \neq 0$, the group $G_A \simeq \mathbb{Z}/4\mathbb{Z}$ acts transitively on the four roots of $P \cdot P^*$:

- *the half turn ρ_A^2 swaps the two roots of each quadratic: $H = \{\mathrm{Id}, \rho_A^2\}$ preserves each quadratic globally;*
- *the quarter turns ρ_A, ρ_A^3 take the real roots to the complex roots, and conversely;*
- *$G_A/H \simeq \mathbb{Z}/2\mathbb{Z}$ swaps the quadratic and its dual.*

No rotation of order 2 for this passage

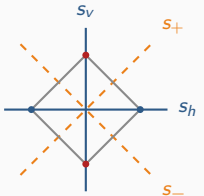
"Preserve": the set of roots, not each root. The only element of order 2 is ρ_A^2, which preserves each quadratic: the sequence $1 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 1$ does not split.

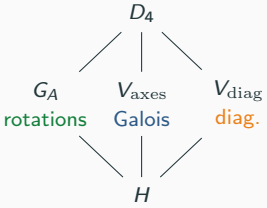
The symmetries of the square

With reflections: the dihedral group D_4, of order 8.

- $s_h : z \mapsto \bar{z}$ and $s_v : z \mapsto 2\mu_a - \bar{z}$ preserve each quadratic globally.
- The diagonals $s_\pm : z \mapsto \mu_a \pm i\,\overline{(z - \mu_a)}$, of order 2, swap real and complex.

Three subgroups of order 4, all containing the half turn: G_A (rotations), V_{axes}, V_{diag}.





The Galois group inside the dihedral group

Rational coefficients, $d = |D|$: the splitting field of $P \cdot P^* = X^4 - d^2$ is $\mathbb{Q}(\sqrt{d}, i)$.

Theorem 10

Through its action on the roots, the Galois group is a subgroup of D_4:

$$\mathrm{Gal} \subset V_{\mathrm{axes}} = \{\mathrm{Id}, \rho_A^2, s_h, s_v\} \subset D_4,$$

with equality if $d \notin \mathbb{Q}^2$, and $\mathrm{Gal} = \{\mathrm{Id}, s_h\}$ if $d \in \mathbb{Q}^2$.

- Galois respects the diagonals (opposite pairs, $x + x' = 2\mu_a$): it lies in D_4.
- It preserves each rational quadratic: it lies in V_{axes}.
- It meets G_A at most in the half turn; the quarter turn is never Galois.

Caution: $x^4 + 4x^2 + 2$ has group $\mathbb{Z}/4\mathbb{Z} \subset D_4$, but its roots lie on the imaginary axis. The D_4 of permutations is geometric only when the roots form a genuine square.

Reading in the theory of the grain

- The quarter turn ρ_A raises the lying grain of P_r into the standing grain of P_c, and lays it down again.
- The half turn flips each grain onto itself.
- The pair (P_r, P_c) is a single grain seen in its four positions: the spinning top of the moving plane, centred at A.
- $\omega = 2^\nu i$ combines a quarter turn with a change of level; ρ_A alone keeps the unit: this is why we get a square, not a rectangle.

The passage from real to complex is a quarter turn from a quadratic to its dual, which exchanges the geometric mean and the quadratic mean.

Reading at two resolutions

Two principles and the imaginary unit

Plane $\Gamma = 2^{-2\nu}\mathbb{Z} + i\,2^{-\nu}\mathbb{Z}$

Proper part

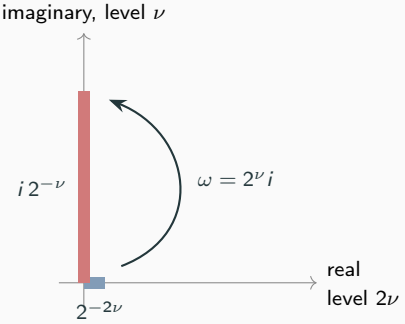
A **refinable coarse grain**: designated as the lower measure of $g_\nu(\alpha)$, made precise at the fine level 2ν.

Partial part

It is **counted in coarse grains** $2^{-\nu}$ on the imaginary axis.

Imaginary unit

$\omega(2^{-2\nu}) = i\,2^{-\nu}$, $\omega = 2^\nu i$: the coarse grain *reached by the quarter turn*.



The cell of a root

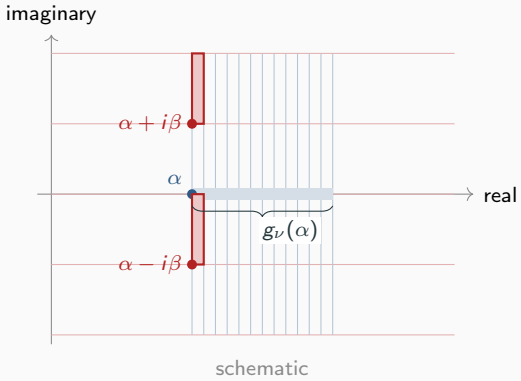
Proposition 11

The root $\alpha + i\beta$ is the corner of the cell

$$C = g_{2\nu}(\alpha) \times i\, g_{\nu}(\beta),$$

fine in width, coarse in height. By promotion, the cell reduces to the standard root.

The lying grain is read at the fine grain; the standing grain is counted in coarse grains.



Three caveats

1. **The ratio of the grains does not come from the quadratic.** The roots and $i^2 = -1$ do not depend on a; the choice $a = 2^\nu$ is that of the reading level.
2. **The asymmetry of precision is a fact of reading.** The gap between the root and its cell is $< 2^{-\nu}$: it vanishes under promotion.
3. **The sign comes from the rotation.** Going from the fine grain to the coarse grain by a reflection would give $j^2 = +1$, and no standing grain.

In Cartesian reading

If the coarse grain is drawn with its length in fine grains (a standard), the roots $(\alpha, \pm a\beta)$ lie on the ellipse $x^2 + y^2/a^2 = \Pi$, an orbit of J.

The analogy with instants and moments

Time	The roots of a quadratic
moment: grain of non-zero duration	grain of the roots, half-spread $\sqrt{\lvert\alpha^2 - \Pi\rvert}$
centre of a moment	proper part $\alpha = \Sigma/2$
instant: moment reduced to a point	double root
junctions of a moment	real roots: endpoints of the lying grain
—	complex roots: endpoints of the standing grain
division of a moment	refinement of the proper part
promotion	the cell reduces to the root

A double root is an instant. Two roots form a grain:
lying if they are real, **standing** if they are complex.

Differential equations and moments

One moment or two moments

$x'' + px' + qx = 0$ on the grid of moments: $X_{m+1} = (I + \Delta A)\, X_m$, $X = (x, x')$.

- **Real**: each mode propagates by $1 + r\Delta$. The moment propagates.
- **Double**: a shear, the pair slides.
- **Complex**: nothing propagates alone; the pair of two successive moments turns by $\theta \simeq \beta\Delta$.



Promotion: the pair turns by βT and its amplitude is multiplied by $e^{\alpha T}$; one recovers $e^{\alpha t}(c \cos \beta t + d \sin \beta t)$.

Two Planck moments?

What can be proved

- $t_{\mathrm{P}} = \hbar / E_{\mathrm{P}}$: the time the Planck quantum takes to turn by **one radian**.
- During t_{P}, a quantum of energy E turns by $\theta = E / E_{\mathrm{P}}$.
- The two-moment structure repeats at every level.

What is ruled out

- $t_{\mathrm{P}} = 2^{-\nu}$ s would force $\nu \approx 144$, finite.
- t_{P} is computed from measured constants: it is standard, not infinitesimal.

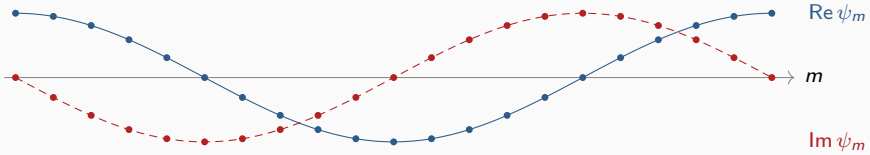
	standard moment $\Delta = t_{\mathrm{P}}$	infinitesimal moment
two Planck moments	literally	in blocks of duration t_{P}
consequence	quasi-energy modulo $2\pi E_{\mathrm{P}}$	no specific prediction

The imaginary reads the next moment

Schrödinger on the grid: $\psi_{m+1} = (1 - i\omega\Delta)\,\psi_m$, $\omega = E/\hbar$.

Proposition 12

$$\operatorname{Im}\psi_m = \frac{\operatorname{Re}\psi_{m+1} - \operatorname{Re}\psi_m}{\omega\,\Delta} \qquad \textit{(exactly).}$$



The imaginary part is, in the present moment, the increment of the real part towards the next moment: the partial part of the next reality.

Imaginaries in the real world

Dirac's fish: a negative number

Three fishermen wake up one after the other; each throws back one fish and takes a third of the rest.

- Expected answer: 25 fish.
- Dirac's answer: -2, the fixed point of $T(x) = \frac{2}{3}(x-1)$.
- All solutions: $N = -2 + 27k$.

Summary

A first-degree equation: its strange solution is **negative**, not imaginary. The example sheds light on negative numbers, not on conjugate roots.

For the passage from individuals to weight to produce an imaginary, one needs **a square** (two passages) and **a reversed return**. Dirac's problem has neither; the following variant has both.

Dirac's fishermen become a fleet

Deviations from equilibrium: stock S_t (tonnes), fishermen F_t (individuals).

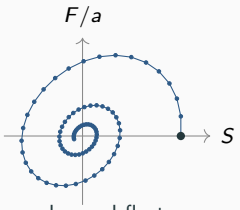
$$S_{t+1} = S_t - h\,F_t, \quad F_{t+1} = F_t + g\,S_t, \quad \lambda = 1 \pm i\sqrt{hg}.$$

g: tonnes $\to$ individuals $(+)$; h: individuals $\to$ tonnes $(-)$.

The square of the passage leads back to the individual: $N^2 = -hg\,I$, dimensionless; partial part $\beta^2 = \Pi - \alpha^2 = hg$.

If fishing added to the stock: $N^2 = +hg\,I$, no cycle.

Real meaning: cycles of abundance and scarcity, $\approx \sqrt{hg}$ radian per season; with departures d, damped if $\frac{d^2}{4} < hg < d$.



damped fleet,
one point per season

Volterra's fish: a genuine imaginary

Prey x, predators y (D'Ancona, Adriatic; Volterra, 1926):

$$x' = \alpha x - \beta xy, \qquad y' = \delta xy - \gamma y.$$

Near equilibrium:

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad \lambda^2 + \alpha\gamma = 0.$$

- Roots $\pm i\sqrt{\alpha\gamma}$: observed cycles of period $2\pi/\sqrt{\alpha\gamma}$.
- The **sign** comes from predation; mutualism would give $j^2 = +1$.
- The matrix is $\sqrt{\alpha\gamma}\, J$, with $a = \sqrt{c/b}$: the **conversion rate** between the two units.

And electricity

In alternating current, reactive power (in var) is the imaginary part of power (in watts for the real part): an imaginary part that bears the name of a unit.

Computational interest

One product fewer

A (discriminant)

$u = 1/a$; $\Delta = b \cdot b - 4(a \cdot c)$;

$x_{\pm} = (-b \pm \sqrt{\Delta}) \cdot \frac{u}{2}$.

B (Viète)

$u = 1/a$; $\alpha = -\frac{b}{2} \cdot u$, $\Pi = c \cdot u$;

$D = \alpha \cdot \alpha - \Pi$; $x_{\pm} = \alpha \pm \sqrt{D}$.

	div.	mult.	roots
A	1	4	1
B	1	3	1

A real but modest gain

A multiplication costs a few cycles; division and square root several times more. The main interest lies elsewhere: **accuracy**.

Loss of digits and the stable algorithm

For $x^2 - 10^8 x + 1$, the small root equals $1.000\,000\,000\,000\,000\,1 \times 10^{-8}$.

Method (double precision)	Small root	Error
classical $(-b - \sqrt{\Delta})/2a$	7.45×10^{-9}	25 %
naive Viète $\alpha - \sqrt{\alpha^2 - \Pi}$	7.45×10^{-9}	25 %
Viète through the product Π/x_1	1.000×10^{-8}	$\sim 10^{-16}$

Taking Viète all the way

$$x_1 = \alpha + \operatorname{sgn}(\alpha)\sqrt{\alpha^2 - \Pi}, \qquad x_2 = \frac{\Pi}{x_1}.$$

Compute the far endpoint of the lying grain, then the near endpoint through the product (Forsythe, Higham).

The fused margin and eigenvalues

A single operation

$D = \mathrm{fma}(\alpha, \alpha, -\Pi)$: a single rounding. $b^2 - 4ac$ needs two products, and Kahan's trick for comparable accuracy.

Limit. Near a double root, an error η on the margin moves the roots by about $\sqrt{|\eta|}$: this is the conditioning of the problem.

2×2 eigenvalues

$$\lambda = \alpha \pm \sqrt{\alpha^2 - \Pi}, \quad \alpha = \tfrac{1}{2} \operatorname{tr} M,\ \Pi = \det M.$$

The quadratic arrives directly in Viète's form, without division. Complex case: modulus $\sqrt{\det M}$.

Conclusion

Conclusion

1. **Origin of the complex plane.** Keep both views of the part: $J = h \circ \tau$, and $J^2 = -I$ because τ is a rotation.
2. **Roots.** $C = \alpha I + \beta K$: homothetic proper part, partial part carried by the rotation.
3. **Viète.** The imaginary measures the excess of the product over the real maximum; hyperbolic or circular according to the sign of the margin.
4. **The grain of the roots.** Lying, point-like or standing; in Γ, the proper part is a refinable coarse grain and the imaginary unit is the coarse grain reached by the quarter turn.
5. **Rotation.** A quarter turn about A takes a quadratic to its dual, real to complex; Galois lies in the dihedral group, never the quarter turn.
6. **Time.** Real roots: the moment propagates; complex roots: two successive moments at once.
7. **Computation.** Using the product gives the stable algorithm.

A double root is an instant; two complex roots form a standing grain.

Thank you

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