Resolution Symmetry of $\widetilde{\mathbb{R}}$

The bidirectional process and the line at infinity

Second volume of the *grain program*

Hugues Genvrin

May 28, 2026

Outline

Recap from Volume I

- Family of carrier circles $(C_k)_{k\in\mathbb{Z}}$, all tangent to Δ at O, radius $\rho_k = 2^k\rho_0$.
- Universal flow Γ with angle doubling $\theta_{k-1} = 2\theta_k$.
- Stratification $\widetilde{\mathbb{N}}_0 \subset \widetilde{\mathbb{N}}_1 \subset \widetilde{\mathbb{N}}_2 \subset \cdots$, with $\widetilde{\mathbb{N}}_k = \{n : 1 \leq n \leq 2^{kn_\infty}\}$.
- Partial field $\widetilde{\mathbb{R}}$ built by deployment at resolution $-2n_\infty$.

The hidden symmetry

Volume I focused on *descending* the levels (refinement of the point O). But the family of circles also extends *upward*: larger and larger circles, dilation of the structure.
What if these two directions were dual?

The geometric process is bidirectional

The angle doubling $\theta_{k-1} = 2\theta_k$ reads in two directions:

- **Downward (refinement)** $\mathcal{F}$: from C_k to C_{k-1} doubles the flow passes through each point. Internal resolution of the origin O.
- **Upward (dilation)** $\mathcal{F}^{-1}$: from C_k to C_{k+1} halves the number of passes; equivalently, doubles the portion of ambient circle needed to host the flow. External resolution of infinity.

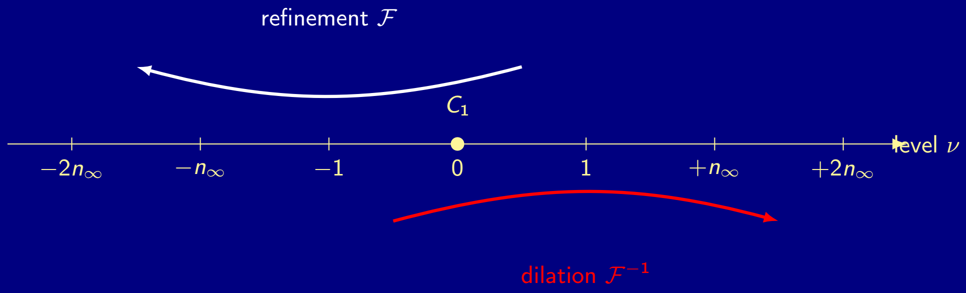
Definition

For $g \in \mathcal{G}_\nu$ with $\nu \in \mathbb{Z}$, $\mathcal{F}^{-1}(g)$ denotes the grain of level $\nu + 1$ of which g is a refinement.

Proposition (Reciprocity)

$\mathcal{F}^{-1}(\mathcal{F}(g)) = g$ and $\mathcal{F}(\mathcal{F}^{-1}(g)) \ni g$.

Bidirectional process, illustrated



Reading.

- Refinement $\mathcal{F}$: from level 1 downward to $-n_\infty$ and beyond, opening up the inside of the point.
- Dilation $\mathcal{F}^{-1}$: from level 1 upward to $+n_\infty$ and beyond, opening up the outside toward infinity.

The ascending axiom

Axiom (Existence of ascending resolutions)

For every $k \in \mathbb{N}^*$, the level $+kn_\infty$ is reached: there exist grains of level $+kn_\infty$, supported on a circle C_{+kn_∞} of radius 2^{kn_∞} and circumference $2\pi \cdot 2^{kn_\infty}$. The curvature is $\kappa_{+kn_\infty} = 2^{-kn_\infty}$.

Reading. This axiom extends Axiom 1 of Volume I (existence of descending resolutions $-kn_\infty$) symmetrically. Both directions of the process reach their limit levels with the same stratified structure.

Curvature: the crucial dichotomy at $+n_\infty$ and $+2n_\infty$

Proposition (Infinitesimal curvature at $+2n_\infty$)

- For $k = 1$: $\kappa_{+n_\infty} = 2^{-n_\infty}$ exceeds the inversion threshold $1/(2\pi \cdot 2^{n_\infty})$. *Standard* invertible curvature.
- For $k \geq 2$: $\kappa_{+kn_\infty} = 2^{-kn_\infty}$ is below the threshold. *Non-invertible infinitesimal* curvature.

Reading.

- At level $+n_\infty$: huge circle but its curvature remains invertible in $\widetilde{\mathbb{R}}$. Effective circular geometry.
- At level $+2n_\infty$: the circle C_{+2n_∞} has curvature *indistinguishable from zero* in the partial field. It *is* a line.

This is the first appearance of the *line at infinity*.

The line at infinity Δ_∞

Definition

The *line at infinity* Δ_∞ is the circle C_{+2n_∞} identified with its tangent at O. This identification is legitimate: the curvature 2^{-2n_∞} is a non-invertible infinitesimal, so C_{+2n_∞} is indistinguishable from a line within the partial field $\widetilde{\mathbb{R}}$.

We deploy $\mathcal{F}^{-n_\infty}$ with a developpement of a zero with width 2π on Δ_∞. **What does Δ_∞ look like?**

- Total length $2\pi \cdot 2^{2n_\infty}$.
- Carries a stratified structure mirroring that of $\widetilde{\mathbb{R}}^+$.

The detailed structure is the central theorem of this volume.

Structure of Δ_∞: the double mosaic

Theorem (Stratified structure of Δ_∞)

Δ_∞ has total length $2\pi \cdot 2^{2n_\infty}$ and carries a *double mosaic*:

- **Coarse level** $(+n_\infty)$: 2^{n_∞} *dilated copies of C_1*, each of length $2\pi \cdot 2^{n_\infty}$.
- **Fine level** $(+2n_\infty)$: 2^{2n_∞} *dilated zeros* (the original g_{-n_∞} grain-points of C_1, seen externally), each of length 2π.

Each coarse copy of C_1 contains 2^{n_∞} fine dilated zeros.

Reading. What was *internal* to C_1 at $-2n_\infty$ (grain-points) reappears *externally* at $+2n_\infty$ (dilated zeros). The two sides are structurally symmetric.

Symmetry of architecture

	Descending side ($-2n_\infty$)	**Ascending side ($+2n_\infty$)**
Coarse level	2^{n_∞} grain-points inside C_1	2^{n_∞} dilated copies of C_1
Fine level	2^{2n_∞} sub-positions of the deployment	2^{2n_∞} dilated zeros
Fine elementary length	$2\pi/2^{n_\infty}$	2π
Total length	$2\pi \cdot 2^{n_\infty}$	$2\pi \cdot 2^{2n_\infty}$

Same architecture, dual length scales. Both sides count the same number of elements (2^{n_∞} at coarse level, 2^{2n_∞} at fine level). What differs is the elementary length: very small inside, very large outside.

Caractéristiques	Couple stratifié	Résolution /Multiplicité	Observations	Résolution /Multiplicité*	Couple stratifié*	Objet géométrique*
$\mathcal{G}_{-n_\infty} = \widehat{O}$	$(2\pi, 2^{n_\infty})$	-n$_\infty / 2^{n_\infty}$	Brisure de symétrie	$+n_\infty / \frac{1}{2^{n_\infty}}$	$(2\pi, 2^{n_\infty})$	$C_{n_\infty}, \kappa > 0$
$\Delta = \widetilde{\mathbb{R}^+}$ $\epsilon_0 = \frac{2\pi}{2^{n_\infty}}$ $\# = 2^{2n_\infty}$ $\mathcal{L} = 2\pi.2^{n_\infty}$	$(4\pi^2, 2^{2n_\infty})$	$-2n_\infty / 2^{2n_\infty}$	Symétrisation	$+2^n_\infty / \frac{1}{2^{2n_\infty}}$	$(4\pi^2, 2^{2n_\infty})$	$\Delta_\infty = C_{2n_\infty}$ $\kappa = 0$ $\# = 2^{n_\infty}$ $\epsilon_0 = 2\pi.2^{n_\infty}$ $\mathcal{L} = 2\pi.2^{2n_\infty}$

Figure: Synthèse

Ascending stratification

Definition (External strata)

$$\widetilde{\mathbb{N}}_k^{\text{ext}} := \{n : 1 \le n \le 2^{kn_\infty}\} \quad (\text{with } \widetilde{\mathbb{N}}_0^{\text{ext}} = \mathbb{N}).$$

$\widetilde{\mathbb{N}}_k^{\text{ext}}$ indexes the external ordinal positions at dilation depth k.

Same cardinal, different semantics. $\widetilde{\mathbb{N}}_k^{\text{ext}}$ and $\widetilde{\mathbb{N}}_k$ are isomorphic as ordered sets (2^{kn_∞} elements, same order structure). They differ only in meaning:

- $\widetilde{\mathbb{N}}_k$: internal ordinal positions (descending resolution).
- $\widetilde{\mathbb{N}}_k^{\text{ext}}$: external copies counts (ascending dilation).

The canonical bijection δ_k

Proposition

There exists a canonical bijection

$$\delta_k : \widetilde{\mathbb{N}}_k \to \widetilde{\mathbb{N}}_k^{\text{ext}}$$

sending the i-th internal ordinal position on C_1 (at level $-kn_\infty$) to the i-th external ordinal copy (at dilation depth $+kn_\infty$). δ_k preserves the order.

The backbone of resolution duality. δ_k is the combinatorial spine relating the two sides. It will be extended in the next part to a geometric bijection between grains.

External arithmetic: cyclic addition

Definition (External mosaic)

$\mathcal{M}_{+kn_\infty}$: the set of 2^{kn_∞} ordinal copies of C_1 rectified on the giant circle C_{+kn_∞} (identified with Δ_∞ for $k \geq 2$).

Proposition (Cyclic structure)

$(\mathcal{M}_{+kn_\infty}, \oplus^{\text{ext}})$ is a finite cyclic abelian group of order 2^{kn_∞}, where $\oplus^{\text{ext}}$ is modular addition on the giant circle.

Reading. External arithmetic is the mirror image of internal arithmetic: instead of adding ordinal positions on C_1, one translates copies on the giant circle.

External multiplication: scale composition

Definition (External multiplication)

For $m_{j_1}^{(k_1)} \in \mathcal{M}_{+k_1 n_\infty}$ and $m_{j_2}^{(k_2)} \in \mathcal{M}_{+k_2 n_\infty}$:

$$m_{j_1}^{(k_1)} \odot^{\text{ext}} m_{j_2}^{(k_2)} := m_{j_1 \cdot j_2}^{(k_1+k_2)}.$$

Stratum climbing.

$$\mathcal{M}_{+k_1 n_\infty} \odot^{\text{ext}} \mathcal{M}_{+k_2 n_\infty} \subset \mathcal{M}_{+(k_1+k_2) n_\infty}.$$

Geometric meaning. Multiplying two copies of C_1 composes their scales: it lands in the finer mosaic at depth $k_1 + k_2$. This mirrors the lexical multiplication of Volume I, which climbed from $\widetilde{\mathbb{N}}_1$ to $\widetilde{\mathbb{N}}_2$.

The main theorem of the volume

Theorem (Resolution duality, geometric version)

For every $k \geq 1$, the canonical bijection $\delta_k : \mathcal{G}_{-kn_\infty} \to \mathcal{M}_{+kn_\infty}$ satisfies:

1. **Preservation of addition.** $\delta_k(g_{i_1} \oplus g_{i_2}) = \delta_k(g_{i_1}) \oplus^{\text{ext}} \delta_k(g_{i_2})$.
2. **Compatibility with stratifying multiplication.** $\delta_{k_1+k_2}(g_{i_1} \odot g_{i_2}) = \delta_{k_1}(g_{i_1}) \odot^{\text{ext}} \delta_{k_2}(g_{i_2})$.

Reading. The internal multiplication $\odot$ contracts fine positions while descending the stratification. The external multiplication $\odot^{\text{ext}}$ dilates coarse copies while ascending. δ_k *interleaves* these two movements: they are two faces of the same underlying operation.

Canonical values and stratified couples

We have an invariant as a product of finesses. At $+2n_\infty$ the fine elementary lenght is $\frac{2\pi}{2^{n_\infty}}$ and at the coarse level the elementary lenght is $2\pi 2^{n_\infty}$. The product is invariant. We refind this invariant in a self-similarity approach approach.

Theorem (Canonical values)

For every $k \in \mathbb{N}^*$,

$$\ell_k = (2\pi)^k \quad \text{and} \quad \mu_k = 2^{kn_\infty}.$$

Stratified couples. Four invariant couples emerge:

- Resolution $-n_\infty$: $f_1 = (2\pi, 2^{n_\infty})$.
- Resolution $-2n_\infty$: $f_2 = (4\pi^2, 2^{2n_\infty})$.
- Resolution $+n_\infty$: $f_1^\star = (2\pi, 2^{n_\infty})$.
- Resolution $+2n_\infty$: $f_2^\star = (4\pi^2, 2^{2n_\infty})$.

Same couples, different sides. The numerical content is identical between starred and non-starred sides. The duality is a *relativity of magnitudes* according to the chosen factorization.

Self-similarity

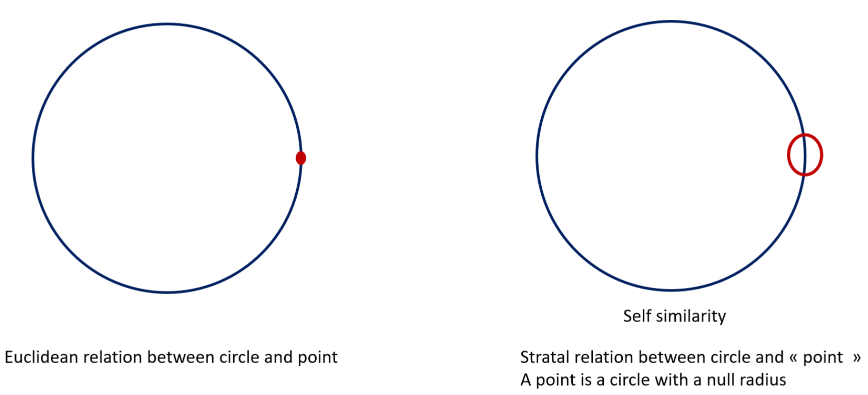


Figure: Self-similarity versus Euclidean approach

The invariant as self-similarity exponent

A second reading of $(2\pi)^2$. The invariant $(2\pi)^2$ admits an interpretation in terms of *multiplicative self-similarity* of the grain process.

Stratal re-parametrization of C_1. The reference circle C_1 at resolution $-kn_\infty$ admits an equivalent factorization where the elementary thickness is measured in stratal units rather than Euclidean units:

$$(\mathcal{L}_{\text{stratal}}, |\mathcal{L}|, \epsilon_{\text{stratal}}) = \left((2\pi)^k,\ 2^{kn_\infty},\ \frac{(2\pi)^k}{2^{kn_\infty}}\right).$$

This is the *same circle* as the standard factorization $(2\pi, 2^{kn_\infty}, 2\pi/2^{kn_\infty})$, but with elementary thickness integrating cumulative stratal factors over the k resolution levels traversed.

Self-similarity exponent

The factor 2π between adjacent strata is a *characteristic exponent* of the program's multiplicative self-similarity. Each strata transition multiplies stratal length by 2π (the circumference of C_1).

What this self-similarity is — and is not

What it *is*.

- A structural *multiplicative invariance of scale*: each strata transition is generated by the same factor 2π.
- A *characteristic exponent* that organizes the hierarchy of stratal couples: $(2\pi, 2^{n_\infty})$, $((2\pi)^2, 2^{2n_\infty})$, etc.
- The structural origin of the invariant $(2\pi)^2$ encountered previously: it is the *square* of the basic stratal factor, corresponding to the crossing of two strata.

What it is *not*.

- A fractal dimension in the Hausdorff sense: 2π is not a non-integer dimension, it is a multiplicative scaling exponent.
- A topological dimension change: the underlying geometric support remains C_1 (one-dimensional).

Reading. The invariant $(2\pi)^2$ is recovered through this self-similarity: it is the square of the elementary stratal factor between adjacent levels, expressing how the geometric circle is « seen » through its successive stratal re-parametrizations.

The dictionary in one table

Feature	**Internal side** $(-kn_\infty)$	**External side** $(+kn_\infty)$
Geometric object	Grain-point $\mathcal{G}_{-kn_\infty}$	Copy $\mathcal{M}_{+kn_\infty}$
Cardinal	2^{kn_∞}	2^{kn_∞}
Addition	On C_1	On C_{+kn_∞}
Multiplication	$\widetilde{\mathbb{N}}_a \cdot \widetilde{\mathbb{N}}_b \subset \widetilde{\mathbb{N}}_{a+b}$	$\mathcal{M}_{+an_\infty} \cdot \mathcal{M}_{+bn_\infty} \subset \mathcal{M}_{+(a+b)n_\infty}$
Stratifying	Descending	Ascending

Reading. The bijection δ_k relates these two columns coherently. The arithmetic on one side mirrors the arithmetic on the other, with the stratification reversed in direction.

The factorization triplet

Theorem (Factorization)

Every grain g, deployed at resolution $\nu \in \{-2n_\infty, -n_\infty, +n_\infty, +2n_\infty\}$, is characterized by a triplet $(\mathcal{L}, |\mathcal{L}|, \epsilon)$:

- $\mathcal{L}$: total length of the deployment;
- $|\mathcal{L}|$: cardinality of elementary elements;
- ϵ: length of one elementary element (the « thickness of zero »).

The three quantities satisfy the *fundamental relation*

$$\boxed{\mathcal{L} = |\mathcal{L}| \cdot \epsilon.}$$

Reference triplet at resolution $-n_\infty$ of C_1:

$$(\mathcal{L}, |\mathcal{L}|, \epsilon) = \left(2\pi,\ 2^{n_\infty},\ \frac{2\pi}{2^{n_\infty}}\right).$$

The four canonical factorizations

Resolution	$\mathcal{L}$	$\lvert\mathcal{L}\rvert$	ϵ	Geometric object
$-n_\infty$	2π	2^{n_∞}	$\frac{2\pi}{2^{n_\infty}}$	C_1, base resolution
$-2n_\infty$	$2\pi \cdot 2^{n_\infty}$	2^{2n_∞}	$\frac{2\pi}{2^{n_\infty}}$	Deployed line $\widetilde{\mathbb{R}}^+$
$+n_\infty$	$2\pi \cdot 2^{2n_\infty}$	2^{n_∞}	$2\pi \cdot 2^{n_\infty}$	Δ_∞, coarse copies of C_1
$+2n_\infty$	$2\pi \cdot 2^{2n_\infty}$	2^{2n_∞}	2π	Δ_∞, dilated zeros

All triplets satisfy $\mathcal{L} = \lvert\mathcal{L}\rvert \cdot \epsilon$. The four canonical factorizations are not separate objects: they are four ways of looking at the *same underlying substrate* from different resolution levels.

Relativity of magnitudes

Corollary (Relativity of magnitudes)

The infinite and the finite are *relative to the chosen factorization* : what is elementary at one resolution becomes total at another, and vice versa.
Every factorization must lead to an equilibrium of the quantities involved through the relation $\mathcal{L} = |\mathcal{L}| \cdot \epsilon$. This yields an *equivalence of forms* according to the resolution level.

The deep statement

The infinite reduces to the finite, and vice versa.

What this volume accomplishes

1. **Bidirectional process.** The geometric process is *intrinsically* bidirectional: refinement and dilation are symmetric.
2. **The line at infinity.** $\Delta_\infty = C_{+2n_\infty}$, first level where curvature becomes non-invertible infinitesimal.
3. **Double mosaic on Δ_∞.** 2^{n_∞} coarse dilated copies of C_1 + 2^{2n_∞} fine dilated zeros. Mirror image of the descending deployment.
4. **The invariant $(2\pi)^2$.** Crossed product of finenesses, the true standard invariant of the duality, dimensional in nature.
5. **Canonical bijection δ_k.** Order-preserving, compatible with both additive and multiplicative structures. The combinatorial backbone of duality.
6. **Factorisation**. Magnitudes are relatives.

Outlook

- **Volume III — Divergent integrals.** The double mosaic provides the natural setting for stratified Riemann sums. Every divergent integral becomes a value in some $\widetilde{\mathbb{R}}_{|k}$. Classical divergences classify as strata.
- **Volume IV (forthcoming) — Algebraic auto-duality.** The bijection δ_k is the combinatorial face. The algebraic face is the involution $\iota : X \mapsto 1/X$ on $\widetilde{\mathbb{R}}^*$. Relating the two requires a precise theorem of realization that we leave for further work.
- **Possible directions.**
  - Topological consequences of self-duality (self-dual plane, line $XY = 1$, fundamental half-plane).
  - Structural analogies with T-duality, IR/UV duality, non-commutative geometry. Suggestive, not yet derivative.

Thank you.

Questions?

hugues@genvrin.fr

Hugues Genvrin

Programme du grain

May 28, 2026