

Set Theory : What Promotion Erases

Reflexivity, the Axiom of Choice and Dedekind Cuts
in Grain Theory

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Overview

Three studies, one thesis.

1. **Reflexivity** — a part in bijection with the whole
2. **The axiom of choice** — choosing an element in each set
3. **Dedekind cuts** — building the reals from their gaps

In each case, the hyperfinite grid gives a **sharp, canonical** answer:

counting · well-ordering · discreteness

and **promotion** $\tilde{\mathbb{R}}_{\text{fin}}/\mathfrak{m} = \mathbb{R}$ transforms it:

equal cardinals · need for choice · completeness

Grain theory: reminders

I. The paradox of reflexivity

II. The axiom of choice

III. Dedekind cuts

Synthesis

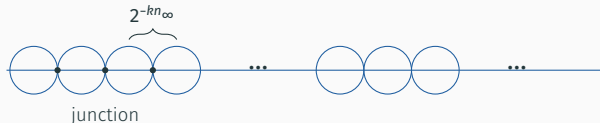
Grain theory: reminders

Grids, grains and junctions

Let n_∞ be a hyperinteger (unlimited integer). At resolution $-N$, the unit continuum $[0, 1[$ is split into 2^N **grains**

$$g_\ell^N = [\ell 2^{-N}, (\ell + 1) 2^{-N}[, \quad 0 \leq \ell < 2^N.$$

- The points $\ell 2^{-N}$ are the **junctions**; two junctions $\ell 2^{-N}, (\ell + 1) 2^{-N}$ are **adjacent**.
- Grain-theory grids: resolutions $-kn_\infty$, grains of infinitesimal thickness 2^{-kn_∞} .
- The grid B_{-kn_∞} is **hyperfinite**: $|B_{-kn_\infty}| = 2^{kn_\infty}$.
- Grains are ordered **lexically** (binary words of length kn_∞).



Promotion and transfer

Promotion

The passage to the quotient $\tilde{\mathbb{R}}_{\text{fin}}/\mathfrak{m} = \mathbb{R}$, where $\mathfrak{m}$ is the ideal of infinitesimals, realized by the **standard part st.**

- all grain thicknesses 2^{-kn_∞} collapse to 0;
- the grid cardinal 2^{kn_∞} becomes the continuum $2^{\aleph_0}$.

Theorem 1 (Transfer)

First-order statements true for all integers (resp. finite sets) remain true for all hyperintegers (resp. hyperfinite sets and their **internal** subsets).

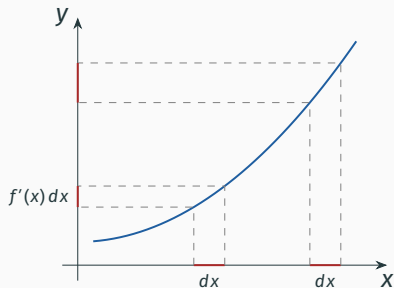
Internal subsets

Subsets definable at the level of the grid. “Unlimited” is *not* internal: it refers to the standard world, i.e. to what promotion sees.

Cartesian anomaly: functional stratification

Theorem 2 (Functional stratification)

A differentiable f maps the uniform interval $[x, x + dx]$ of the Ox axis onto $[f(x), f(x) + f'(x) dx]$ of the Oy axis. It acts on its codomain axis as a **local stratification factor** $|f'(x)|$.



For a homothety $x \mapsto ax$, the factor is constant and equals a .

I. The paradox of reflexivity

The paradox

- Galileo (1638): squares $\leftrightarrow$ all integers.
- Bolzano (1851): $x \mapsto 2x$ maps $[0, 1]$ onto $[0, 2]$.
- Dedekind (1888): a set is infinite iff equipotent to a proper part. Russell: such classes are reflexive.

If $F \subset E$ are two continua (non-empty interior):

$$|F| = |E| = 2^{\aleph_0}.$$

Euclid's axiom "*the whole is greater than the part*" fails for cardinals.

Question

What happens at a given floor of the tower of hyperfinite grids?

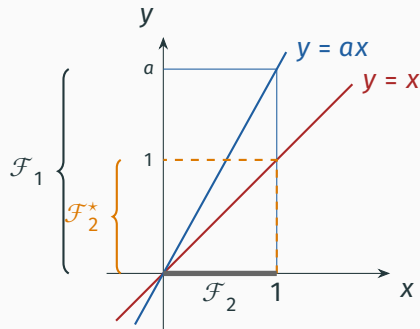
Setup

Let $a > 1$ be standard.

- $\mathcal{F}_2 = [0, 1[$ on Ox , with 2^{n_∞} grains of thickness 2^{-n_∞} ;
- $f(x) = ax$, $\mathcal{F}_1 = f(\mathcal{F}_2) = [0, a[$ on Oy ;
- $\tau = \text{rot}(O, \pi/2)$, $\mathcal{F}_2^* = \tau(\mathcal{F}_2) = [0, 1[$ on Oy .

Then $\mathcal{F}_2^* \subsetneq \mathcal{F}_1$:

$\mathcal{F}_2^*$ is the **part**, $\mathcal{F}_1$ the **whole**.



Two grids

The part: $\mathcal{F}_2^* = \tau(\mathcal{F}_2)$

τ is an isometry:

2^{n_∞} grains of thickness 2^{-n_∞} .

Homogeneous dyadic grid.

The whole: $\mathcal{F}_1 = f(\mathcal{F}_2)$

Stratification factor a :

2^{n_∞} grains of thickness $a 2^{-n_\infty}$.

Homothetic grid.

Proposition 3 (Heterogeneous bijection)

$h = f \circ \tau^{-1} : \mathcal{F}_2^* \rightarrow \mathcal{F}_1, y \mapsto ay$, is a bijection mapping grain onto grain. Both sets have 2^{n_∞} grains.

The part is in bijection with the whole, even at the hyperfinite level,
but h dilates every grain by the Jacobian factor a .

Counting at a homogeneous resolution

Proposition 4 (Homogeneous counting)

Counted in dyadic grains of thickness 2^{-n_∞} :

$$N(\mathcal{F}_2^*) = 2^{n_\infty}, \quad N(\mathcal{F}_1) = \lfloor a 2^{n_\infty} \rfloor,$$

$$N(\mathcal{F}_1) - N(\mathcal{F}_2^*) \geq (a - 1) 2^{n_\infty} - 1 \quad \text{is unlimited.}$$

Corollary 5 (No homogeneous bijection)

No bijection $\mathcal{F}_2^* \rightarrow \mathcal{F}_1$ maps each dyadic grain onto a dyadic grain of the same thickness (hyperfinite pigeonhole principle).

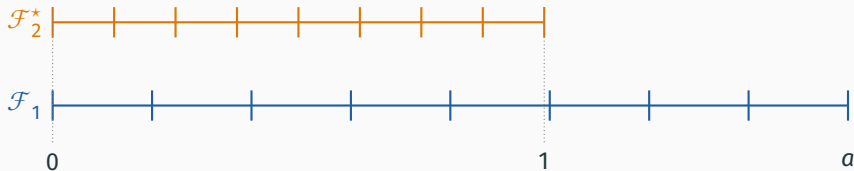
Remark 6 (Ratio)

$\text{st}(N(\mathcal{F}_2^*)/N(\mathcal{F}_1)) = 1/a$: the grain count encodes what Lebesgue measure keeps and cardinality loses.

Irrational factor: incommensurable grids

Proposition 7

If a is a standard irrational, the grain boundaries of $\mathcal{F}_1$ and $\mathcal{F}_2^*$ share only the point 0:
 $ak2^{-n_\infty} = m2^{-n_\infty} \Rightarrow k = m = 0$ (by transfer).



The inclusion $\mathcal{F}_2^* \subset \mathcal{F}_1$ is an inclusion of **points**, not of **grids**: “part/whole” needs a common resolution.

Resurgence through promotion

Theorem 8 (Resurgence)

After promotion, the grids become $[0, 1[$ and $[0, a[$, and

$$|[0, 1[| = |[0, a[| = 2^{N_0}.$$

The heterogeneous bijection h becomes Bolzano's map $y \mapsto ay$.

- $\text{st}(2^{-n_\infty}) = \text{st}(a 2^{-n_\infty}) = 0$: the thickness difference is erased;
- the unlimited gap in grain counts disappears with the grains;
- 2^{N_0} absorbs the factor a .

Thesis I

Reflexivity is not a property of the floors of the tower but an **effect of promotion**.

Related programme: numerosities of Benci and Di Nasso, which restore Euclid's principle by labelling and finite approximation.

II. The axiom of choice

The axiom

Axiom of choice (Zermelo, 1908)

If T is a set of non-empty, pairwise disjoint sets, their union contains a set S having exactly one element in common with each element of T .

- Equivalent to the **well-ordering theorem** (Zermelo, 1904).
- On a well-ordered set, choice is trivial: take the least element.
- Grids are well-ordered by the lexical order.

Question

Can grains justify the axiom of choice?

Finite and hyperfinite choice

Proposition 9 (Finite choice)

If $T = \{T_1, \dots, T_n\}$ with n standard, T has a choice function. This is a **theorem of ZF** (induction on n): no axiom needed.

Corollary 10 (Hyperfinite choice)

If n is a hyperinteger and $(T_i)_{i \leq n}$ is internal, it has an internal choice function (by transfer).

Extracting one grain from each T_i is *correct*, but it proves something that does **not** depend on the axiom of choice.

The axiom is needed only for *infinite* (standard) families of arbitrary subsets of a set that cannot be well-ordered.

A canonical choice: the first grain

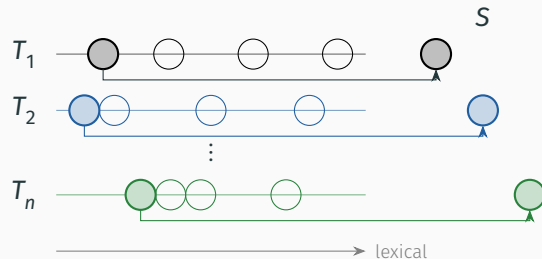
Theorem 11 (Canonical choice)

For an internal family (T_i) of non-empty sets of grains,

$$c(T_i) = \min T_i$$

(first grain in lexical order) is a choice function.

Grains let us choose **without arbitrariness**:
Zermelo's mechanism reversed.



The internal hypothesis is necessary

$E = \{g_m : m \text{ unlimited}\}$ is non-empty but has no least element: if m is unlimited, so is $m - 1$.

Choose, then promote

For a standard $A \subset [0, 1[$, let $\mathcal{G}(A) = \{g : g \cap {}^*A \neq \emptyset\}$ and $\sigma(A) = \text{st}(\min \mathcal{G}(A))$.

Proposition 12

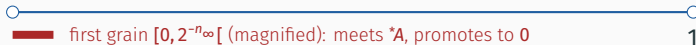
For every standard non-empty $A \subset [0, 1[$: $\sigma(A) = \inf A$.

Example 13 (Failure on an open set)

$A =]0, 1[$: the first grain is $[0, 2^{-n_\infty}[$, which meets *A ; but $\sigma(A) = 0 \notin A$.

$$\sigma(A) = 0 \notin A$$

$$A =]0, 1[$$



Open sets: the second grain, or the first interior grain

Example 14 (The second grain of $]0, 1[$)

The second grain $g_1 = [2^{-n_\infty}, 22^{-n_\infty}[$ has representative $2^{-n_\infty} \in {}^*A$: before promotion, it gives a point **inside** the set.

Remark 15 (A fixed rank does not work)

$A = \{x\}$: no second grain. $A = \{0\} \cup]1/2, 1[$: the second grain is $[1/2, 1/2 + 2^{-n_\infty}[$, with representative $1/2 \notin A$.

Proposition 16 (First interior grain)

Let $\mathcal{G}^0(A) = \{g : g \subset {}^*A\}$. If A has non-empty interior, $\mathcal{G}^0(A) \neq \emptyset$ and the representative of $\min \mathcal{G}^0(A)$ lies in *A . For $]0, 1[$, $\min \mathcal{G}^0(A) = g_1$: **the second grain**.

But promotion erases it again

Every grain g_m of limited rank promotes to $\text{st}(m 2^{-n_\infty}) = 0 \notin]0, 1[$.

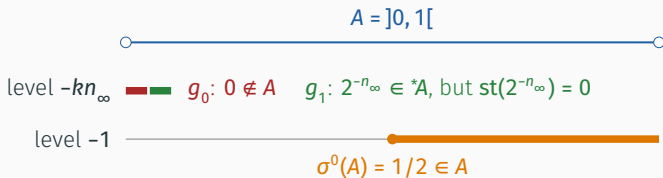
Choosing at the coarsest level

Theorem 17 (Coarse choice)

For $A \subset [0, 1[$ with non-empty interior, let $N(A)$ be the least standard N such that a grain of resolution $-N$ lies in A , and $\ell(A)$ the least index of such a grain. Then

$$\sigma^0(A) = \ell(A) 2^{-N(A)} \in A$$

is a standard dyadic: a choice function **invariant under promotion**, and **definable in ZF**.



A grain version of the classical ZF procedure: take the first rational of a fixed enumeration of $\mathbb{Q}$ lying in the open set.

The frontier: sets with empty interior

Proposition 18

If A has empty interior, $\mathcal{G}^0(A) = \emptyset$ at every resolution, standard or hyperfinite (by transfer, *A contains no interval, even infinitesimal).

Subsets of $[0, 1[$	Canonical grain choice	Survives st
closed	first grain of $\mathcal{G}(A)$	yes (= $\min A$)
non-empty interior	first interior grain	no
non-empty interior	first grain at the coarsest level	yes (dyadic)
empty interior, not closed	none	—

Vitali classes $x + \mathbb{Q}$ are countable, hence have **empty interior**.

Where grains give a choice that survives promotion, it is already definable in ZF. The frontier of the axiom of choice runs through the sets with **empty interior**, where no grain fits entirely in the set.

What survives, and Vitali

Corollary 19

σ is a choice function on non-empty subsets of $[0, 1[$ closed in $[0, 1[$, where it coincides with $A \mapsto \min A$.

The canonical choice survives exactly where it adds nothing:

$A \mapsto \min A$ on closed sets is **already definable in ZF**.

Vitali

The classes of $\mathbb{R}/\mathbb{Q}$ are dense: $\mathcal{G}(A)$ is the whole grid for each class, and σ picks 0 for all of them. The canonical procedure does not distinguish the classes.

The continuum is not definably well-orderable in ZF: the grain does not create this well-order; it moves it to the grid, where it does not survive **st**.

Foundations

- Usual nonstandard models (ultrapowers) use the **ultrafilter lemma**, a weak form of choice.
- Nelson's IST is conservative over **ZFC**.
- Hrbacek and Katz: infinitesimal analysis **without** choice.

Open point

Whether the canonical choice holds in ZF alone depends on how the axiomatics of grain theory introduces n_∞ .

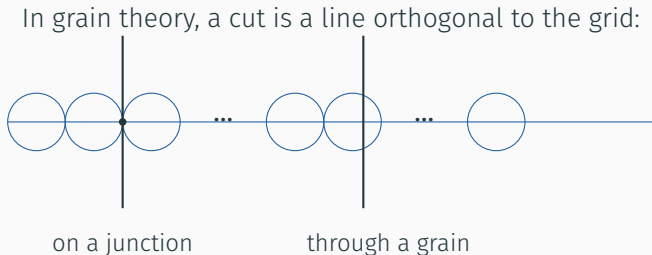
Thesis II

On grids, choice is **canonical**; promotion destroys this canonicity. The axiom of choice is the price paid, on the continuum, to restore by postulate the order that promotion erased.

III. Dedekind cuts

Cuts on a grid

- **Dedekind** (1872): a real is a cut (A, B) of $\mathbb{Q}$; if no rational lies at the boundary, the cut reveals a **gap**.
- **Cantor** (1872): reals as limits of nested segments.

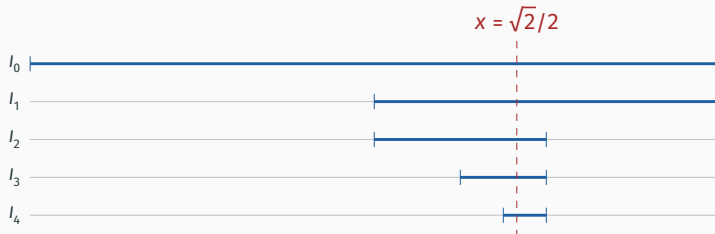


Natural guess: junction = rational, grain = irrational. **This is false.**

Nested adjacent dyadic segments

Theorem 20 (Nested segments)

For $x \in [0, 1[$, $I_N(x) = [\ell_N 2^{-N}, (\ell_N + 1) 2^{-N}]$ with $\ell_N = \lfloor 2^N x \rfloor$: endpoints are adjacent dyadics; $I_{N+1} \subset I_N$; $\bigcap_N I_N(x) = \{x\}$.



Corollary 21 (Irrationals)

Every irrational is the limit of nested segments with adjacent dyadic endpoints; it is interior to each, and the sequence is unique (a dyadic such as $1/2$ has two: $0,1000 \dots = 0,0111 \dots$).

Grains as nested segments

- $I_N(x)$ is (up to its right end) the **grain** containing x .
- Left or right half at each step = binary digit $b_{N+1} = \ell_{N+1} - 2\ell_N$.
- By transfer, the grains containing x at resolutions $-n_\infty, -2n_\infty, -3n_\infty, \dots$ are nested, with adjacent junctions as endpoints (cf. the construction of π).

Proposition 22 (One level suffices)

For N unlimited:

$$\text{st}(\ell_N 2^{-N}) = \text{st}((\ell_N + 1)2^{-N}) = x.$$

Cantor's limit $N \rightarrow \infty$ is replaced by **one unlimited resolution followed by promotion**.

Junction or grain: the dyadic dichotomy

Proposition 23 (The case of $1/3$)

$1/3$ is a junction of no grid, standard or hyperfinite: $1/3 = \ell 2^{-N} \Rightarrow 3\ell = 2^N$, impossible (transfer).

- p/q (reduced) is a junction $\iff q$ is a power of 2.
- Junction/grain separates **dyadics** from non-dyadics, not rationals from irrationals.

Proposition 24 (Junctions promote to any real)

For N unlimited, every $x \in [0, 1[$ is the promotion of a junction of the grid of resolution $-N$.

The junction nearest to $\sqrt{2}/2$ is a dyadic hyperrational whose promotion is irrational.

The exact criterion: position inside the grain

Relative position of x in its grain: $\theta_N(x) = \text{frac}(2^N x) \in [0, 1[$ (0: left junction, $\frac{1}{2}$: middle).

Lemma 25 (Doubling)

$\theta_{N+1}(x) = \text{frac}(2 \theta_N(x))$: halving the grain $\leftrightarrow$ doubling map.

Theorem 26 (Rationality criterion)

1. x dyadic $\iff \theta_N(x) = 0$ eventually;
2. x rational $\iff (\theta_N(x))_N$ takes finitely many values;
3. x irrational $\iff (\theta_N(x))_N$ takes infinitely many values.

Proof of 2 ($\Leftarrow$): $\theta_{N+T} = \theta_N \Rightarrow x = m / (2^N(2^T - 1))$.

Example: $1/3 = 0,010101 \dots$

$$\theta_N(1/3) = \begin{cases} 1/3 & N \text{ even} \\ 2/3 & N \text{ odd} \end{cases}$$

grain, $N = 0$ |-----●-----| $\theta = 1/3$

grain, $N = 1$ |-----●-----| $\theta = 2/3$

grain, $N = 2$ |-----●-----| $\theta = 1/3$

grain, $N = 3$ |-----●-----| $\theta = 2/3$

The cut always crosses a grain, but always at one third or two thirds:
a rational cut through a grain. For $\sqrt{2}/2$, all θ_N differ.

A quasi-hole of the grid, not of $\mathbb{R}$

Relative gap: $\delta_N(x) = \min(\theta_N(x), 1 - \theta_N(x))$.

Proposition 27 (Rationals keep their distance)

For p/q non-dyadic and every N (integer or hyperinteger):

$$\delta_N(p/q) \geq \frac{1}{q}, \quad \left| \frac{p}{q} - \frac{\ell}{2^N} \right| \geq \frac{2^{-N}}{q}.$$

- $\delta_N(1/3) = 1/3$: the grid never comes closer than a third of a grain.
- **No hole in $\mathbb{R}$** : both junctions of the grain promote to **1/3**.
- For the grid, **1/3** is not a hole: it is a **grain**.

Remark 28 (The converse fails)

Thue–Morse constant **0,0110100110010110 ...**: irrational, never three equal digits in a row, hence $\delta_N \geq 1/8$ for all N .

A symmetry, not a symmetry breaking

Lemma 29 (Reflection)

For non-dyadic x : $\theta_N(1 - x) = 1 - \theta_N(x)$.

- $x \mapsto 1 - x$ swaps $1/3$ and $2/3$; the orbit $\{1/3, 2/3\}$ is a 2-cycle invariant under reflection.
- Promotion erases θ : it makes things **more** symmetric.

Proposition 30 (Dependence on n_∞)

Only levels $N = kn_\infty$ matter, and n_∞ is even or odd:

- n_∞ even: $\theta_{kn_\infty}(1/3) = 1/3$ for all k ;
- n_∞ odd: $\theta_{kn_\infty}(1/3)$ alternates with the parity of k .

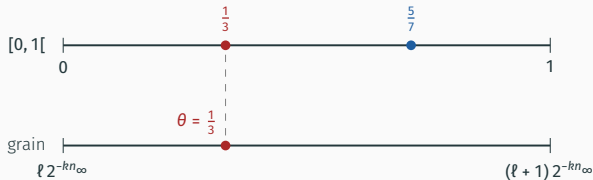
If anything is “broken”, it is by the **choice of n_∞** , not by the real line.

The factorial choice: a self-similar tower

Theorem 31 (Self-similarity of rationals)

If $n_\infty = H!$ (H unlimited), then for every standard p/q with q odd and every $k \geq 1$:
 $\theta_{kn_\infty}(p/q) = \text{frac}(p/q)$.

Proof: the order d of 2 mod q is standard, so $d \mid H!$ and $q \mid 2^{kn_\infty} - 1$; then $2^{kn_\infty} \frac{p}{q} = \frac{p}{q} + p \frac{2^{kn_\infty} - 1}{q}$.



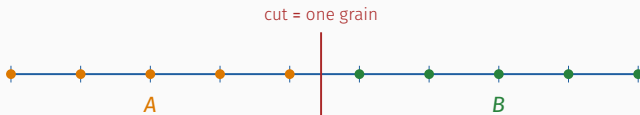
Proposition 32

If $\theta_N(x) = x$ for a standard x , then x is rational with odd denominator: **irrationals are never self-similar**.

Grids have no holes

Proposition 33 (No holes)

Every internal cut (A, B) of the junctions of a hyperfinite grid is determined by two adjacent junctions: $\max A = \ell 2^{-N}$, $\min B = (\ell + 1) 2^{-N}$.



For irrational x , the grid cut between $\ell_N 2^{-N}$ and $(\ell_N + 1) 2^{-N}$ is perfectly determined; promotion identifies its two edges into a point that is not rational.

Thesis III

The completeness of Dedekind's continuum is the image, under promotion, of the discreteness of the grids.

Synthesis

One pattern, three instances

	Grid level	After promotion
Reflexivity	part < whole at homogeneous grain (thickness 2^{-n_∞} vs. $a 2^{-n_\infty}$)	$ F = E = 2^{N_0}$ (thicknesses collapse to 0)
Axiom of choice	canonical choice $\min T_i$ (well-order; coarsest level on open sets)	choose-then-promote = $\inf$ (axiom needed for empty interior)
Dedekind cuts	no holes; cut = adjacent junctions (discreteness)	gaps of $\mathbb{Q}$, completeness (edges identified)

Common thesis

What is **sharp and canonical** on the hyperfinite grids — counting, well-order, discreteness — is exactly what **promotion erases**. The classical “paradoxes” and postulates of the continuum are the shadow of this erasure.

Open questions

1. **Reflexivity:** replace $x \mapsto ax$ by any f with $|f'| > 1$; homogeneous counting should involve $\int |f'|$ and the logarithmic stratal metric.
2. **Numerosities:** does homogeneous grain counting coincide with a numerosity in the sense of Benci–Di Nasso?
3. **Foundations:** can n_∞ be introduced without any form of choice, so that canonical choice holds in ZF?
4. **Choice of n_∞ :** can the axiomatics allow the factorial choice $n_\infty = H!$, making the tower self-similar for all rationals?

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