Squaring the circle

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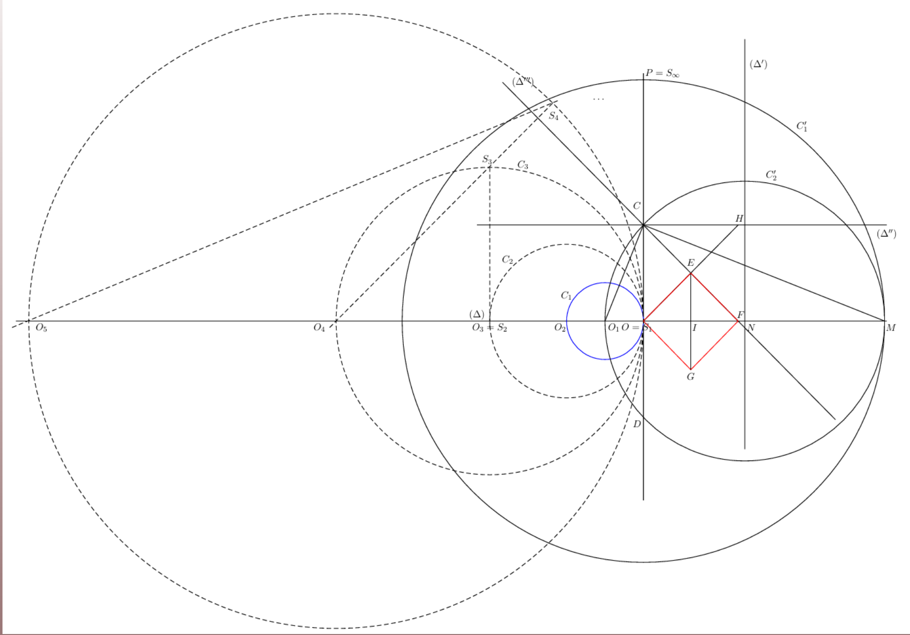
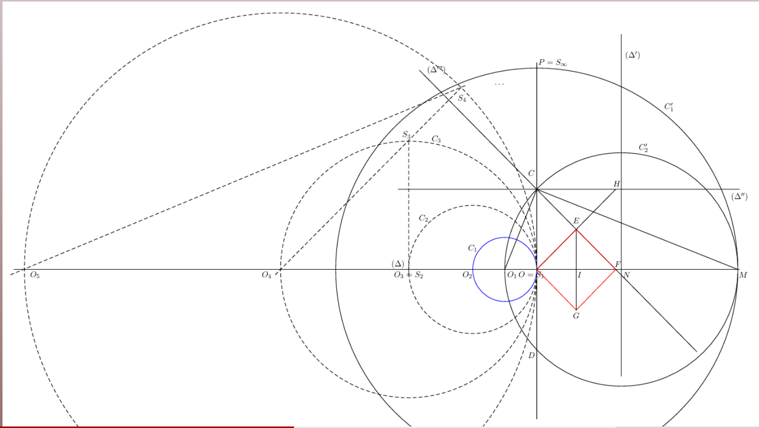

$P = S_\infty$
(Δ')
(Δ''')
S_4
C'_1
S_3
C_3
C'_2
C
H
(Δ'')
E
C_2
C_1
F
(Δ)
O_5
O_4
$O_3 = S_2$
O_2
O_1
$O = S_1$
I
N
M
G
D

Rectification

Starting the construction

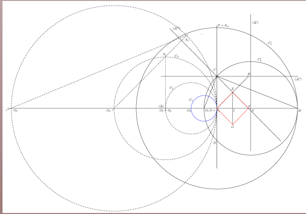
We consider a sequence of circles : $(C_{n,n\in N})$ with $O_{n,n\in N}$ as centers and $\rho_{n,n\in N}$ as radius, where $\rho_n = 2^{n-1}\rho_1(\rho_1 \in R^{+*})$. We plot the three first points : $S_1 = O, S_2 = O_3, S_3$.



Rectification

Recursive process

If we draw (O_{n+1}, S_n) (we remind that S_n is on C_n), the straightline will intercept C_{n+1} and $S'_{n+1} = O_{n+1}$. S_{n+1} is the point from the upper half plane of (Δ). O_{n+1}, S_n, S_{n+1} are aligned, so $\widehat{OO_{n+1}S_n} = \widehat{OO_{n+1}S_{n+1}}$. On the other hand, by theorem, the center angle $\widehat{OO_nS_n}$ is a double value of the angle to the extremity of the diameter from $\widehat{OO_{n+1}S_n}$.

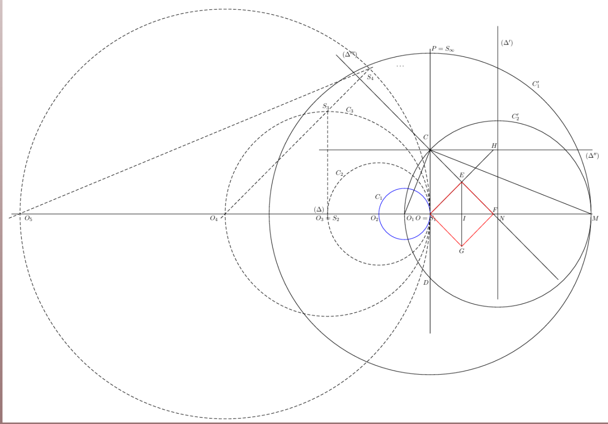


Rectification

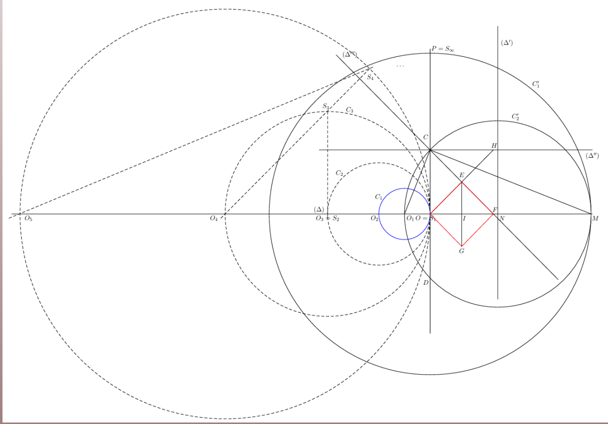
We can conclude that the curvilinear arc (OS_n) tends toward $[O, P]$, orthogonal to Δ, with $2\pi\rho_1$ as lenght. Thus, $[O, P]$ form the rectification of C_1.
The bundle of circles arcs (O, S_n) define a whole achieved. We conclude that $[OS_n]$ goes to its bound, when n tends to infinity. We show that we can have the rectification in an infinite countable steps with only the rule and the compass.

Squaring the circle

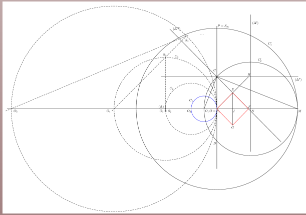
With $[O;P]$ definated, we consider the circle $C'1$ with radius equals to $[O;P]$ and center O. It intercept (OO_1) on 2 points. We call M the point such as $O \in [O_1, M]$



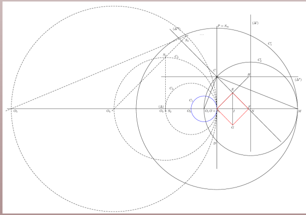
We determine N the middle of $[O_1, M]$ by drawing the mediator of $[O_1, M]$: (Δ'). The mediator of a segment could be obviously draw by the only rule and compass.



We draw the circle C_2' with center N, and radius $[N, M]$. The circle intercepts $[O, P]$ on C and D. We call C the point belong to $[O, P]$. The triangle (O_1, C, M) is inscribed in the circle C_2' with diameter $[O_1, M]$. Hence this triangle is a right angle on C. We can notice that $(OC) \perp (MO_1)$. So the triangles (O_1, O, C) and (N, O, C) are rectangle on O. By the theorem of Pythagore, we check $OC^2 = OO_1.OM$. As $OM = OP = 2\pi\rho_1$ and $O_1O = \rho_1$, we have $OC = \sqrt{2\pi}\rho_1$.



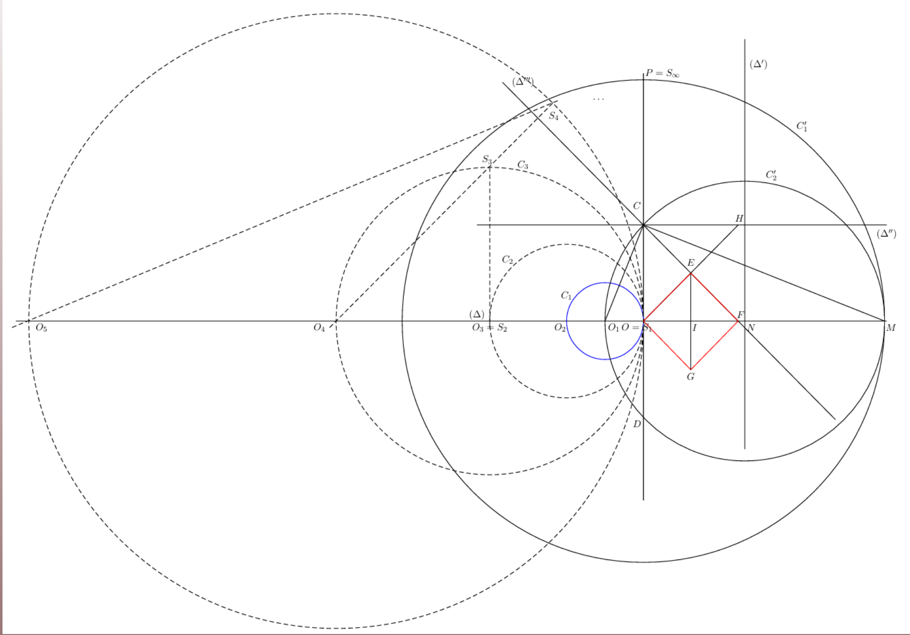
We draw up the perpendicular to (OC) on C, that we call (Δ''), obviously determinable with only the rule and the compass. We put H such as $CH = CO$, we report the magnitude with the compass. Even, we draw the triangle (O, C, H) which is isoscele and right on C.



We build with the compass the the point F on Δ such as $OE = OF$. We can build the image point from E to G by the mediator, with the intersection of two circles well choiced (we are keeping the magnitude of compass $[O, E]$) with center O and F. As the triangle (O, C, H) is isoscele with $\widehat{OCH} = \frac{\pi}{2}$, then $\widehat{COE} = \frac{\pi}{4}$. In consequence, $\widehat{OEG} = \frac{\pi}{4}$ and as (E, O, G) is isoscele $\widehat{OGE} = \frac{\pi}{4}$, so $\widehat{EOG} = \frac{\pi}{2}$,

By the definition of the mediators, we have $(OF) \bigcap (EG) = I$, middle of $[OF]$.
So, the diagonal of the quadrilateral (O, E, F, G) are cutting in the middle, we have (O, E, F, G) as a parallelogram. With the precedent results, we deduct that $OE = EF = FG = GO$ that imply that the parallelogram is a rhombus. Otherwise we have $\widehat{EOG} = \frac{\pi}{2}$. So one angle of the parallelogram is right, then (O, E, F, G) is a rectangle, so we conclude that (O, E, F, G) is a square, with a side equals to $\sqrt{\pi}\rho_1$. So, the area of the square is equal to $\pi\rho_1^2$, which is the surface of C_1.

Squaring the circle is accomplished with only the rule and the compass in $n_\infty + 7$ steps.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