Divergent Integrals and Regularization by Stratification

Third volume of the *grain program*

Hugues Genvrin

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Outline

The grain program in three volumes

1. **Axiomatique du grain** (Volume I): construction of the partial field $\widetilde{\mathbb{R}}$ from a geometric process (circles with angle doubling); recovery of $\mathbb{R}$ as standard restriction.
2. **Symétrie de résolution** (Volume II): the geometric process is bidirectional. Downward refinement ($\mathcal{F}$) and upward dilation ($\mathcal{F}^{-1}$) are dual. The double mosaic on Δ_∞. Canonical bijection δ_k.
3. **Divergent integrals** (Volume III, this talk): application to divergent integrals. The natural setting for a stratified regularization.

Central idea of Volume III

A Riemann sum on the double mosaic is *always finite* in the self-engendered hierarchy. Classical divergences become values in higher strata; the projection $\widetilde{\mathbb{R}} \to \mathbb{R}$ acts as intrinsic regularization.

Notation reminders

- n_∞: limit resolution level (idealized symbol added to $\mathbb{Z}$).
- $L_\infty := 2\pi \cdot 2^{n_\infty}$: total length of the deployment $\widetilde{\mathbb{R}}^+$.
- $\Delta x := 2\pi/2^{n_\infty}$: fine elementary length on the mosaic.
- $\widetilde{\mathbb{R}}_{|k}$: stratum k, set of elements with absolute value $\leq 2^{kn_\infty}$.
- st: standard part projection.
- Strict stratification convention: $c \cdot 2^{kn_\infty}$ with c standard > 1 does *not* lie in $\widetilde{\mathbb{R}}_{|k}$ but in $\widetilde{\mathbb{R}}_{|k+1}$.

The double mosaic of $\widetilde{\mathbb{R}}^+$

$\widetilde{\mathbb{R}}^+$ is the deployment of C_1 at resolution $-2n_\infty$, segment $[0, L_\infty[$ equipped with two levels of stratification:

- **Coarse level** $(-n_\infty)$: 2^{n_∞} ordinal positions, each of length 2π.
- **Fine level** $(-2n_\infty)$: 2^{2n_∞} sub-positions, each of fineness $\Delta x = 2\pi/2^{n_\infty}$.

We write

$$x_{N,j} := (N-1)\cdot 2\pi + j \cdot \Delta x$$

for $N \in \widetilde{\mathbb{N}}_1$, $j \in \widetilde{\mathbb{N}}_1$, and $\mathcal{X} := \{x_{N,j}\}$ for the set of fine sub-positions.

Definition of stratified integration

Definition

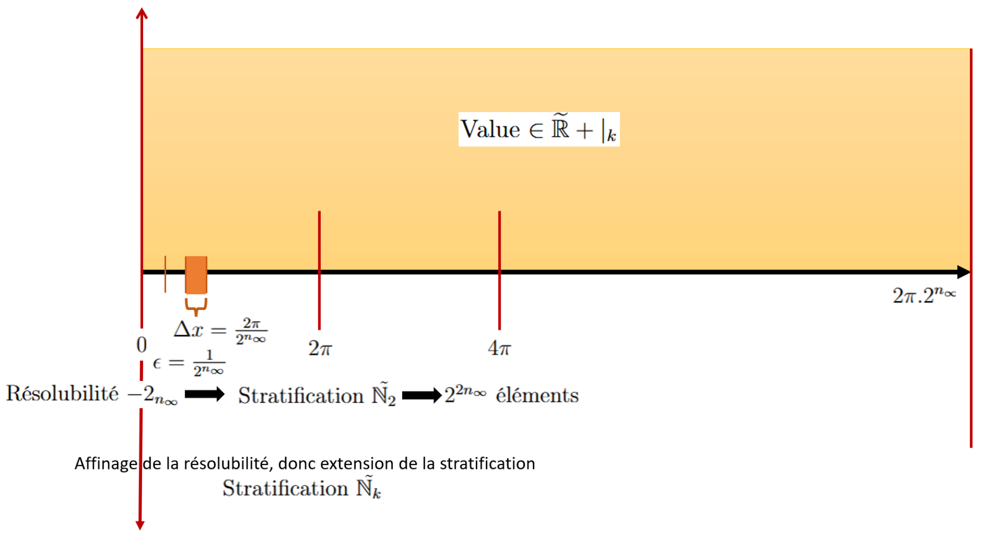
For f admissible and $[a, b]$ in $\widetilde{\mathbb{R}}$ with a, b mosaic positions:

$$\widetilde{\int_a^b} f(x)\,dx \;:=\; \sum_{x_{N,j} \in [a,b[} f(x_{N,j}) \cdot \Delta x.$$

Key feature. The sum has at most 2^{2n_∞} terms; it is always finite as an element of a high-enough stratum of the self-engendered hierarchy.

No need for limits. Classical divergence becomes a value in some $\widetilde{\mathbb{R}}_{|k}$; the question is only *which stratum*.

Topological support of integration



Résolubilité -2_{n_∞} → Stratification $\tilde{\mathbb{N}}_2$ → 2^{2n_∞} éléments

Affinage de la résolubilité, donc extension de la stratification

Stratification $\tilde{\mathbb{N}}_k$

Compatibility with the classical Riemann integral

Theorem (Standard compatibility)

For f continuous and Riemann-integrable on a standard interval $[a, b] \subset \mathbb{R}$:

$$\widetilde{\int}_a^b f(x)\,dx \;=\; \int_a^b f(x)\,dx \;+\; \varepsilon,$$

where ε is infinitesimal. In particular, $\mathrm{st}\left(\widetilde{\int}_a^b f\right) = \int_a^b f$.

Proof idea. Classical estimate $|S_{\Delta x}(f) - \int_a^b f| \leq C\Delta x(b-a)$, valid for continuous f. Here Δx is infinitesimal (but invertible: the ratio $\Delta x/(1/L_\infty) = (2\pi)^2 \approx 39.5$ is well above the inversion threshold).

First examples

Example 1: $\widetilde{\int}_0^{L_\infty} 1\,dx$

2^{2n_∞} terms each equal to Δx, summing to $L_\infty = 2\pi \cdot 2^{n_\infty}$.
Prefactor $2\pi > 1$, so $L_\infty \in \widetilde{\mathbb{R}}_{|2}$ (not $\widetilde{\mathbb{R}}_{|1}$).

Example 2: $\widetilde{\int}_0^{L_\infty} x\,dx$

By Faulhaber and direct computation:

$$\widetilde{\int}_0^{L_\infty} x\,dx \sim \frac{L_\infty^2}{2} = 2\pi^2 \cdot 2^{2n_\infty}.$$

Prefactor $2\pi^2 \approx 19.74 > 1$, so result lies in $\widetilde{\mathbb{R}}_{|3}$.

The regularization stratum $\sigma(f)$

Definition

For f admissible and positive,

$$\sigma(f) := \min \left\{ k \in \mathbb{N} \ : \ \widetilde{\int_0^{L_\infty}} f \in \widetilde{\mathbb{R}}_{|k} \right\}.$$

Reading. $\sigma(f)$ measures the *stratal resource* required to host the integral of f.

Convention. The name « regularization stratum » (rather than « divergence stratum ») is positive: it indicates the depth of the counter-term to bring the value back to $\mathbb{R}$.

Polynomial divergences

Theorem

For $p \in \mathbb{N}^*$ *standard,*

$$\widetilde{\int}_0^{L_\infty} x^p \, dx = \frac{L_\infty^{p+1}}{p+1} + \varepsilon_p = \frac{(2\pi)^{p+1}}{p+1} \cdot 2^{(p+1)n_\infty} + \varepsilon_p.$$

Prefactor $c_p := (2\pi)^{p+1}/(p+1)$ *is standard,* > 1 *for* $p \geq 1$. *Hence*

$$\boxed{\sigma(x^p) = p + 2.}$$

Note. The shift $\sigma(x^p) = p + 2$ (not $p + 1$ as a naive order-of-magnitude reading would suggest) reflects that the standard analytic prefactor consumes an extra stratum.

Inverse (UV) divergences

Theorem

For $q \in \mathbb{N}^*$ standard,

$$\widetilde{\int}_{\Delta x}^{1} \frac{dx}{x^q} \sim \begin{cases} \zeta(q) \cdot 2^{(q-1)n_\infty} / (2\pi)^{q-1} & \text{if } q \geq 2, \\ n_\infty \ln 2 + O(1) & \text{if } q = 1. \end{cases}$$

Prefactors are < 1 in all cases. Hence

$$\sigma(1/x^q) = q - 1 \text{ for } q \geq 2, \quad \sigma(1/x) = 1.$$

UV-IR symmetry. The substitution $u = 1/x$ exchanges divergences at the origin (UV) and at infinity (IR). Prefactors < 1 vs > 1 naturally distinguish them.

Logarithmic divergences

Theorem

For $k \in \mathbb{N}$ standard,

$$\widetilde{\int}_1^{L_\infty} (\ln x)^k \, dx \sim (2\pi)(n_\infty \ln 2)^k \cdot 2^{n_\infty}.$$

Global prefactor $2\pi(\ln 2)^k$ standard; factor n_∞^k remains sub-stratal for k standard. Hence

$$\boxed{\sigma((\ln x)^k) = 2.}$$

Reading. All powers of the logarithm sit in $\widetilde{\mathbb{R}}_{|2}$ regardless of k: the logarithm is *too slow* to climb the hierarchy on its own.

Exponential divergences

Theorem

For $c > 0$ standard,

$$\widetilde{\int}_0^{L_\infty} e^{cx}\,dx \sim \frac{e^{cL_\infty}}{c} = \frac{2^{\alpha \cdot 2^{n_\infty}}}{c}, \quad \alpha = \frac{c \cdot 2\pi}{\ln 2}.$$

The exponent $\alpha \cdot 2^{n_\infty}$ exceeds kn_∞ for every standard k. Hence

$$\boxed{\sigma(e^{cx}) = \infty.}$$

The exponential leaves $\widetilde{\mathbb{R}}$. It lives in the extension $\widetilde{\widetilde{\mathbb{R}}}$ (and beyond): the self-engendered hierarchy naturally hosts what classical analysis cannot.

Summary table of strata

Function	**Integral**	**Stratum $\sigma(f)$**
x^p $(p \geq 1)$	$\sim L_\infty^{p+1}/(p+1)$	$p+2$
$1/x^q$ $(q \geq 2)$	$\sim \zeta(q) \cdot 2^{(q-1)n_\infty}$	$q-1$
$1/x$	$\sim n_\infty \ln 2$	1
$(\ln x)^k$	$\sim (2\pi) n_\infty^k 2^{n_\infty}$	2
e^{cx}	$\sim 2^{\alpha 2^{n_\infty}}$	∞

Hierarchy reading. Polynomial and logarithmic functions stay inside $\widetilde{\mathbb{R}}$ (finite strata). Inverse functions also do (UV-IR duality). The exponential jumps to $\widetilde{\widetilde{\mathbb{R}}}$ at once.

Regularization by projection

Definition

For f admissible and $\widetilde{\int} f \in \widetilde{\mathbb{R}}_{|k}$ with $k \geq 1$,

$$\mathrm{Reg}_\sigma\left(\int f\right) := \mathrm{st}\left(\frac{\widetilde{\int}_0^{L_\infty} f(x)\,dx}{\lambda}\right),$$

where $\lambda \in \widetilde{\mathbb{R}}_{|k}$ is a *counter-term* chosen so that the ratio falls in $\mathbb{R}$.

Division, not subtraction. The counter-term is divided out, extracting the leading coefficient. This is closer in spirit to zeta-regularization than to additive renormalization schemes.

Polynomial regularization

Theorem

For $p \in \mathbb{N}^*$ standard and $\lambda_p := L_\infty^{p+1}/(p+1)$,

$$\mathrm{Reg}_\sigma\left(\int_0^\infty x^p\,dx\right) = 1.$$

Proof. $\widetilde{\int}_0^{L_\infty} x^p = \lambda_p + \varepsilon_p$ with ε_p/λ_p infinitesimal. So $\widetilde{\int}/\lambda_p = 1 +$ infinitesimal, standard part $= 1$.

Geometric reading. The counter-term λ_p is exactly the dominant asymptotic term. The regularized value is the *normalized leading coefficient*, here 1.

Logarithmic regularization

Theorem

$\mathrm{Reg}_\sigma\left(\int_0^1 dx/x\right) = 1$ *with counter-term* $\lambda = n_\infty \ln 2$*.*

Proof. $\tilde{\int}_{\Delta x}^{1} dx/x = n_\infty \ln 2 + O(1)$. Divided by λ: $1 + O(1)/(n_\infty \ln 2)$. The ratio is infinitesimal, standard part $= 1$.

Universal reading. Both polynomial and logarithmic regularizations give 1. The value 1 is the universal normalized leading term. The function-specific information lies in the *sub-leading corrections*.

σ as a function of the upper bound

Definition

For $f \geq 0$ admissible and B a mosaic position,

$$\sigma(f, B) := \min \left\{ k : \overset{\sim}{\int}_0^B f \in \widetilde{\mathbb{R}}_{|k} \right\}.$$

$\sigma(f) = \sigma(f, L_\infty)$ is the maximal-bound case.

Critical bound

$$B_k(f) := \sup\{B : \sigma(f, B) \leq k\}.$$

Idea. Instead of asking « in which stratum does the integral sit? », ask « up to where can we integrate while staying in stratum k? ».

Critical bounds for the main classes

Theorem

For $f(x) = e^x$,

$$B_k(e^x) = k \cdot n_\infty \ln 2 \quad \textit{for all } k \geq 1.$$

Proof. $\int_0^B e^x\,dx = e^B - 1$. At $B = c \cdot n_\infty \ln 2$ with c standard positive: $e^B = 2^{cn_\infty}$. Strict stratum: $2^{cn_\infty} \in \widetilde{\mathbb{R}}_{|k}$ iff $c \leq k$.

Conceptual gain. Classical analysis says $\int_0^\infty e^x = \infty$. Our framework refines: for any $B \leq n_\infty \ln 2$, the integral is finite in $\widetilde{\mathbb{R}}$. Beyond, it climbs into higher extensions according to $B/(n_\infty \ln 2)$.

Fine classification by critical bounds

Theorem

- *Polynomial, logarithmic:* B_k *saturates at* L_∞ *from some finite stratum on (depending on the standard prefactor).*
- *Exponential:* $B_k(e^{cx}) = kn_\infty \ln 2/c$*, linear in* k*.*
- *Double-exponential:* $B_k(e^{e^x}) \sim \ln(kn_\infty)$*, logarithmic in* k*.*

Reading. The critical bound B_k measures the *growth rate* of the function:

- Slow growth ⇔ B_k quickly saturates at L_∞.
- Fast growth ⇔ B_k stays small for all k.

Universal regularization at the critical bound

Definition

For f admissible positive and $k \in \mathbb{N}^*$ with $B_k(f) > 0$,

$$\mathrm{Reg}_k(f) := \mathrm{st}\left(\frac{1}{2^{kn_\infty}} \widetilde{\int_0^{B_k(f)}} f(x)\, dx \right).$$

Universal counter-term. Here 2^{kn_∞}, the exact upper bound of $\widetilde{\mathbb{R}}_{|k}$. This counter-term is universal: independent of f. The adaptation to f happens through $B_k(f)$.

Universal regularization theorem

Theorem

For every admissible positive f and every $k \in \mathbb{N}^*$ with $B_k(f)$ non-trivial,

$$\boxed{\mathrm{Reg}_k(f) = 1.}$$

Proof sketch. At the critical bound, $\overset{\sim}{\int}_0^{B_k(f)} f = 2^{kn_\infty} - \varepsilon_k$ where ε_k has strictly lower stratum. Divided by 2^{kn_∞}: $1 - \varepsilon_k / 2^{kn_\infty}$, ratio infinitesimal, standard part $= 1$.

Information lies in sub-dominants

Universality. $\mathrm{Reg}_k(f) = 1$ for every f. This is not a coincidence: the construction normalizes by design.

Where is the function-specific information? In the sub-leading corrections:

$$\frac{\varepsilon_k}{2^{kn_\infty}} \quad \text{viewed as an infinitesimal of } \widetilde{\mathbb{R}}.$$

Analogy with dimensional regularization. In dimensional reg, $1/\epsilon$ is universal but the coefficients of ϵ^{-n} carry the physics. Here the leading value is 1 but the sub-dominants distinguish functions. Systematic development of these sub-dominants would be the stratified analog of ϵ-expansion.

Example: exponential at critical bound

Corollary

For $f(x) = e^x$ *with* $B_1(e^x) = n_\infty \ln 2$*:*

$$\frac{\widetilde{\int}_0^{n_\infty \ln 2} e^x \, dx}{2^{n_\infty}} = \frac{2^{n_\infty} - 1}{2^{n_\infty}} = 1 - \frac{1}{2^{n_\infty}}.$$

So $\mathrm{Reg}_1(e^x) = 1$*, with sub-dominant* $-1/2^{n_\infty}$*.*

Reading. The standard part is 1 (universal). The sub-dominant $-1/2^{n_\infty}$ is an infinitesimal that encodes the specifically exponential behavior of f: namely, the integral falls short of the stratum boundary by exactly one unit out of 2^{n_∞}.

What this volume accomplishes

1. **Stratified Riemann integral** $\widetilde{\int}$, always finite by construction, compatible with classical integration on standard intervals.
2. **Hierarchy of divergences** via the regularization stratum $\sigma(f)$: a clear classification of all standard function classes.
3. **Strict stratification convention**: prefactors > 1 consume a stratum. Precise placement of polynomial, logarithmic, inverse, exponential divergences.
4. **Two regularization schemes**:
   - Reg_σ: tailored counter-term, extracts the leading coefficient (usually $= 1$).
   - Reg_k: universal counter-term, adaptation through the critical bound $B_k(f)$.
5. **Sub-dominants as the carriers of information**: analog of ϵ-expansion in dimensional regularization.

Open directions

- **Algebraic auto-duality of $\widetilde{\mathbb{R}}$**: the involution $\iota : X \mapsto 1/X$ and its precise relation to the geometric duality δ_k. Subject for a fourth volume.
- **Renormalization correspondences**: making the analogies with Pauli-Villars, dimensional regularization, and ζ-regularization rigorous via explicit dictionaries.
- **Applications to physics**.

Thank you.

Questions?
hugues@genvrin.fr

Hugues Genvrin
Programme du grain
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