The Grain Axiomatics

Building $\widetilde{\mathbb{R}}$ and $\mathbb{R}$ from a geometric process

First volume of the *grain program*

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May 25, 2026

Outline

The guiding idea

Central thesis

Numbers do not preexist geometry: they are *counts*, and circular geometry with angle doubling is the natural generating mechanism.

What we build, in order.

1. Multiplicities (finite and transfinite), out of a geometric process.
2. The natural numbers $\mathbb{N}$, as smallest induction-closed subset.
3. The linear continuum at limit resolution.
4. A *partial field* $\widetilde{\mathbb{R}}$.
5. The classical field $\mathbb{R}$ as its standard restriction.

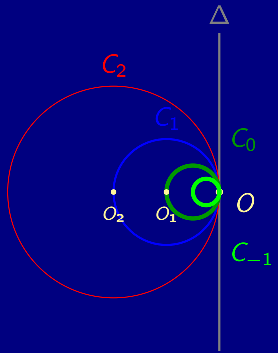
What we do not assume. No prior real number line, no prior set theory beyond the bare minimum. The only « ideal » ingredient is a single limit symbol $-n_\infty$ added to $\mathbb{Z}$.

The four primitives

- **(P1) Euclidean plane E^2** with its usual structure. Fix a point O and a vertical line Δ through O.
- **(P2) Family of carrier circles.** For each $k \in \mathbb{Z}$, the circle C_k of center $O_k = (-2^k \rho_0, 0)$ and radius $\rho_k = 2^k \rho_0$, tangent to Δ at O. We fix $\rho_0 = 1$.
- **(P3) Universal trajectory.** For each $\theta_1 \in [0, 2\pi[$, the trajectory γ_{θ_1} passes through the point of angle $\theta_k = 2^{1-k}\theta_1$ on each C_k. The universal flow is $\Gamma = \{\gamma_{\theta_1}\}$.
- **(P4) Limit-level symbol.** The symbol $-n_\infty$, posited as an ideal level beyond all negative integer levels: $-n_\infty < -n$ for every $n \in \mathbb{N}$.

Remark. (P1)-(P3) are purely geometric. (P4) is the only ideal ingredient: a limit point added to $\mathbb{Z}$, analogous to adjoining ∞ to $\mathbb{R}$ for asymptotic analysis.

The geometry, illustrated



Reading.

- C_2 (radius 2): twice as lare. The flow covers only half the circle.
- C_1 (radius 1): the unit circle. The flow goes around once.
- C_0 (radius 1/2): half the size. The flow goes around twice (angle doubling).

Grains: the geometric atoms

Definition

A *grain* of level ν is a geometric form determined by ν:

- If $\nu = k \leq 1$ in $\mathbb{Z}$: the whole circle C_k, traversed 2^{1-k} times by the flow.
- If $\nu = k \geq 2$ in $\mathbb{Z}$: an arc of C_k (the flow covers only a portion).
- If $\nu = -kn_\infty$: a *geometric point* of C_1 reached at limit resolution.

Notation. $\mathcal{G}_\nu$ for the grains of level ν; $g_0 := C_1$ the initial grain (level 1, multiplicity 1).

Multiplicity and ordinal position

Definition

For a grain g at resolution r with $r \leq \nu(g)$:

- *Multiplicity* $\mu(g, r)$: the number of times the universal flow Γ covers a generic point of g at resolution r.
- *Ordinal position*: the index $i \in \{1, \ldots, \mu(g, r)\}$ of a particular pass.

Theorem (Multiplicity doubling)

For g of level $\nu(g) \leq 1$ and $g' \in \mathcal{F}(g)$ (a refinement, level $\nu(g) - 1$):

$$\mu(g') = 2\mu(g).$$

Geometric reading. Each descent in level doubles the multiplicity because the angle doubling $\theta_{k-1} = 2\theta_k$ makes the flow trace each point twice as often on the next circle.

Stratification of $\widetilde{\mathbb{N}}$

Definition

For $k \in \mathbb{N}$, the *stratum*:

$$\widetilde{\mathbb{N}}_k := \{n : 1 \le n \le 2^{kn_\infty}\} \quad \text{(with } \widetilde{\mathbb{N}}_0 = \mathbb{N}\text{)}.$$

The total set: $\widetilde{\mathbb{N}} = \bigcup_{k \ge 0} \widetilde{\mathbb{N}}_k$.

Filtration.

$$\mathbb{N} = \widetilde{\mathbb{N}}_0 \subset \widetilde{\mathbb{N}}_1 \subset \widetilde{\mathbb{N}}_2 \subset \cdots \subset \widetilde{\mathbb{N}}.$$

Reading. $\widetilde{\mathbb{N}}_k$ is the set of ordinal positions of flow passes on $g_0 = C_1$ at resolution $-kn_\infty$. n_∞ is an ordinary element of $\widetilde{\mathbb{N}}_1$, between finite integers and 2^{n_∞}.

The three axioms

Axiom 1 (Existence of limit resolutions)

For every $k \in \mathbb{N}^*$, the resolution $-kn_\infty$ is reached: $\mu(g_0, -kn_\infty) = 2^{kn_\infty}$.

Axiom 2 (Strict stratification)

For all $k \neq j$, $\widetilde{\mathbb{N}}_k \neq \widetilde{\mathbb{N}}_j$; in particular $2^{kn_\infty} \neq 2^{jn_\infty}$.

Axiom 3 (Total order)

$\widetilde{\mathbb{N}}$ is totally ordered, extending $\mathbb{N}$, with

$$n < n_\infty < 2^{n_\infty} < 2 \cdot 2^{n_\infty} < \cdots < 2^{2n_\infty} < \cdots$$

Minimal commitment. Only three axioms are irreducible. Everything else follows from the geometric process.

Fundamental theorems

Existence of g_∞

The sequence $(\mathcal{F}^n(g_0))_{n\in\mathbb{N}}$ admits as limit a grain g_∞ of level $-n_\infty$ with $\mu(g_\infty) = 2^{n_\infty}$.

Fundamental comparisons

For every $n \in \mathbb{N}$ and $k \geq 1$:

$$n < n_\infty \quad \text{and} \quad kn_\infty < 2^{kn_\infty}.$$

Reading.

- The first inequality follows directly from Axiom 3.
- The second: induction shows $m < 2^m$ for $m \in \mathbb{N}$; passing to the limit, the position kn_∞ stays below the cardinal 2^{kn_∞} that bounds the stratum.

Iterative process and deployment

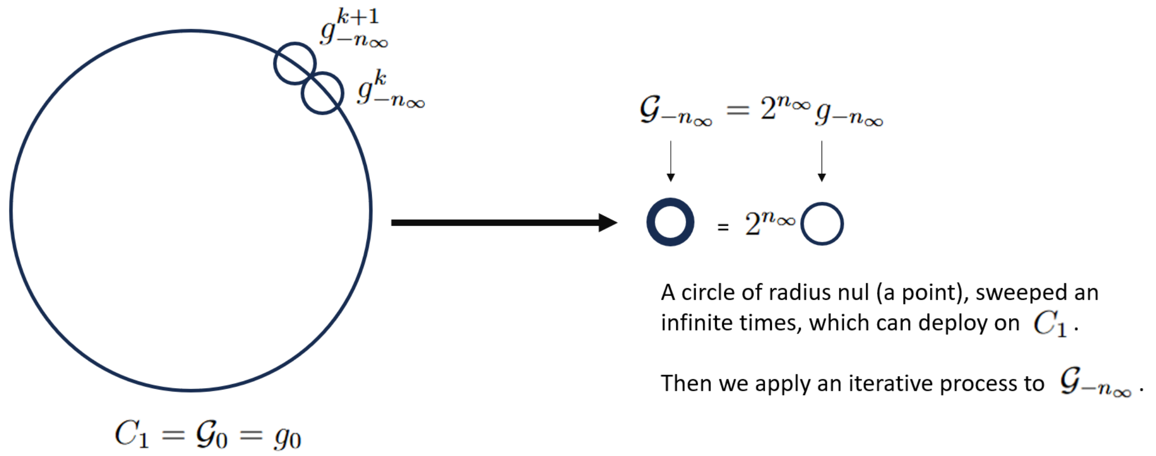


Figure: Deployment

Why we need the arithmetic of positions

The debt. Volume I uses multiplication of positions on C_1 as if it were ordinary real multiplication, without justification.

The plan. Show that position multiplication has a genuine geometric foundation: it is the image, via a ring isomorphism, of the binary multiplication of *lexical writings* of length n_∞.

The strategy.

- Normalize C_1 to a unit-circumference circle C_1' via $h_O^{1/(2\pi)}$.
- Map positions on C_1' to binary words of length n_∞.
- Transport addition and multiplication from binary arithmetic to geometry.
- Return to C_1 by inverse homothety $h_O^{2\pi}$; the scale factor 2π is the price of unfolding.

Binary writings and lexicographic order

Definition

A *binary word* of length n_∞ is a sequence $c = (c_1, c_2, \dots, c_{n_\infty})$ with $c_j \in \{0, 1\}$. We write $\mathcal{B}_{n_\infty}$ for the set of such words.

Lexical writing. To each word c associate $b(c) := 0.c_1 c_2 \cdots c_{n_\infty}$ (binary expansion).

Proposition

$|\mathcal{B}_{n_\infty}| = 2^{n_\infty}$, and $(\mathcal{B}_{n_\infty}, <_{\text{lex}})$ is totally ordered. Order-isomorphic with $\widetilde{\mathbb{N}}_1$.

Reading. The lexicographic order on binary words of length n_∞ is exactly the order on $\widetilde{\mathbb{N}}_1$ given by the rank of the word.

Lexical to geometric isomorphism

Theorem (Order isomorphism)

The correspondence

$$\varphi : \mathcal{B}_{n_\infty} \to \mathcal{G}'_{-n_\infty}, \quad \varphi(b^k) := g'^k_{-n_\infty}$$

is a bijection preserving the order:

$$b^k <_{\text{lex}} b^{k'} \iff g'^k_{-n_\infty} <_{\text{geom}} g'^{k'}_{-n_\infty}.$$

Two transports.

- **Lexical addition** (binary with carry mod 1) transports to modular addition on C'_1 (circumference 1).
- **Lexical multiplication** (ordinary binary product, doubles the length) transports to a multiplication that climbs the stratification: $\widetilde{\mathbb{N}}_1 \times \widetilde{\mathbb{N}}_1 \to \widetilde{\mathbb{N}}_2$. We are in $(\mathcal{B}_{2n_\infty})$.

Multiply : homothetic or decrease in resolution

Twice representation :

- We pass to a resolution $2n_\infty$ on C_1' to adjust the one to one correspondance.
- We make an homothetic transforamtion of the grain.

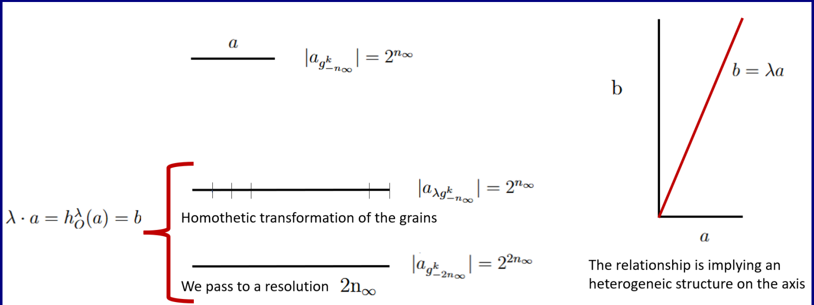


Figure: Multiplication.

Going back to C_1: thickening

Reverse homothety. $h^{-1} = h_O^{2\pi}$ sends C_1' back to C_1, thickening each grain by a factor 2π:

$$g^k_{-n_\infty} = 2\pi \cdot g'^k_{-n_\infty}.$$

Thickened position. To each lexical writing $b^k \in [0, 1[$ associate a position on C_1:

$$x^k := 2\pi \cdot b^k \in [0, 2\pi[.$$

Theorem (Scalar multiplication = homothety)

For $\lambda > 0$ standard, multiplying a position $x \in C_1$ by λ corresponds to the homothety h_O^λ:

$$\lambda \cdot x = h_O^\lambda(x).$$

Geometric meaning. Multiplying a position is uniformly thickening the grain by the factor λ.

Exponentiation

$$e^x = 2^{\frac{1}{\ln 2}.x}$$

None homegenous dilation

$$e^{\pi.2^{n\infty}} > \widetilde{\mathbb{N}}_k, \forall k \in \mathbb{N}$$

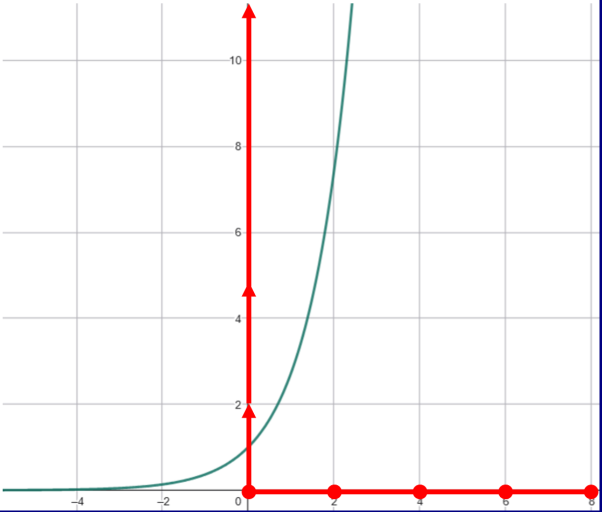


Figure: Exponentiation

Synthesis of position arithmetic

- **Combinatorial foundation.** Multiplication of positions on C_1 is not posited *ad hoc*: it derives from the binary multiplication of lexical writings.
- **Role of homothety.** $h_O^{1/(2\pi)}$ normalizes the circumference to 1, making the lexical-geometric isomorphism immediate. The reverse $h_O^{2\pi}$ gives the scale factor 2π between lexical and positional arithmetic.
- **Stratum climbing.** Multiplication necessarily leaves $\widetilde{\mathbb{N}}_1$ to reach $\widetilde{\mathbb{N}}_2$. This is the combinatorial origin of the need for a deployment at resolution $-2n_\infty$ to host products of positions.
- **Scalar = homothety.** Multiplication by a scalar λ is the homothety of ratio λ: a uniform thickening, which conserve the level of resolution n_∞.

Deployment at resolution $-2n_\infty$

Definition

The deployment of C_1 at resolution $-2n_\infty$: each of the 2^{n_∞} ordinal positions at resolution $-n_\infty$ is split into 2^{n_∞} sub-positions, laid out on a straight line:

$$L_\infty := 2\pi \cdot 2^{n_\infty}.$$

The set $\widetilde{\mathbb{R}}^+$. Positions

$$X = (N-1)\cdot 2\pi + x \quad \text{with } x \in [0, 2\pi[,\ N \in \widetilde{\mathbb{N}}_1.$$

Identified with the segment $[0, L_\infty[$.

Symmetrization. $\widetilde{\mathbb{R}}$ is obtained by adding signs: $\widetilde{\mathbb{R}} = \{(\sigma, X) : \sigma \in \{+,-\},\ X \in [0, L_\infty[\}$ modulo $(+,0) \sim (-,0)$.

Additive structure

Definition (Addition on $\widetilde{\mathbb{R}}^+$)

For $X_1 = (N_1 - 1) \cdot 2\pi + x_1$ and $X_2 = (N_2 - 1) \cdot 2\pi + x_2$:

- If $x_1 + x_2 < 2\pi$: $X_1 + X_2 = (N_1 + N_2 - 2) \cdot 2\pi + (x_1 + x_2)$.
- If $x_1 + x_2 \geq 2\pi$: $X_1 + X_2 = (N_1 + N_2 - 1) \cdot 2\pi + (x_1 + x_2 - 2\pi)$ (with carry).

Defined iff the result remains in $[0, L_\infty[$.

Theorem (Partial abelian group)

$(\widetilde{\mathbb{R}}, +)$ is a *partial abelian group*: commutative, associative (where defined), with identity 0 and opposite $-X$ for every X.

The addition is partial: not defined if the sum exceeds L_∞ in absolute value.

Multiplicative structure

Definition

For $X, Y \in \widetilde{\mathbb{R}}$ with signs σ_X, σ_Y and absolute values $|X|, |Y|$:

$$X \cdot Y := (\sigma_X \cdot \sigma_Y,\ |X| \cdot |Y|), \quad \text{defined iff } |X| \cdot |Y| < L_\infty.$$

The product $|X| \cdot |Y|$ uses the position arithmetic of the previous part.

Definition (Partial multiplicative inverse)

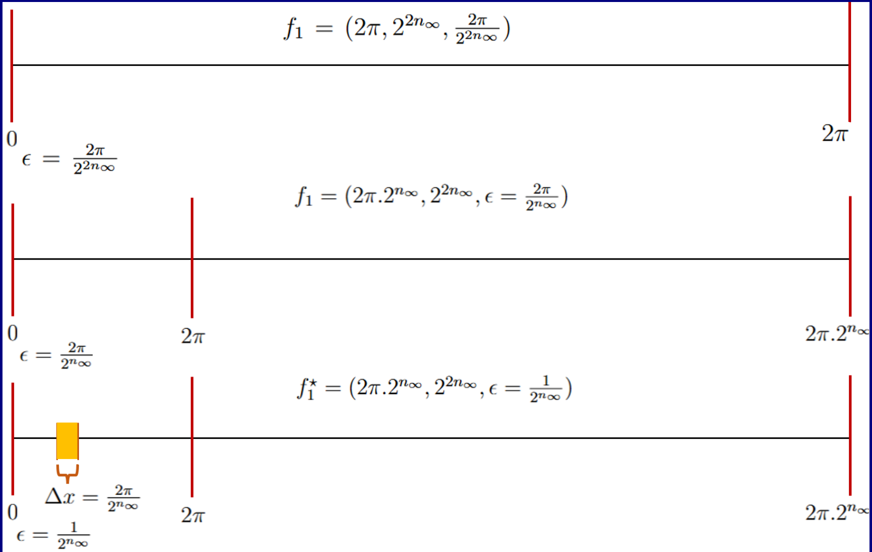
For $X \neq 0$:

$$X^{-1} := (\sigma_X,\ 1/|X|).$$

X^{-1} exists in $\widetilde{\mathbb{R}}$ iff $|X| > 1/L_\infty$.

Non-invertible infinitesimals. Elements with $0 < |X| \leq 1/L_\infty$ exist as positions but have no inverse in $\widetilde{\mathbb{R}}$.

Topology of the extended straightline



The partial field structure

Main theorem

$(\widetilde{\mathbb{R}}, +, \cdot)$ is a *commutative partial field*:

1. $(\widetilde{\mathbb{R}}, +)$ is a partial abelian group.
2. $(\widetilde{\mathbb{R}} \setminus \{0\}, \cdot)$ is a commutative, associative partial monoid with identity 1.
3. Every $X \neq 0$ with $|X| > 1/L_\infty$ admits an inverse X^{-1}.
4. Multiplication is distributive over addition (where all expressions are defined).

Why « partial ». Two reasons:

- Sums larger than L_∞ in absolute value are not defined.
- Elements too small (below $1/L_\infty$) have no multiplicative inverse.

The bounds are not arbitrary: they are the natural thresholds set by the deployment at $-2n_\infty$.

Standard elements

Definition

$X \in \widetilde{\mathbb{R}}$ is *standard* if:

- (bounded) there exists $m \in \mathbb{N}$ with $|X| < 2^m$;
- (non-infinitesimal) $X = 0$, or there exists $m \in \mathbb{N}$ with $|X| > 2^{-m}$.

We write $\mathbb{R} \subset \widetilde{\mathbb{R}}$ for the set of standard elements.

Reading. Standard elements are those that stay *strictly inside* the invertibility zone: bounded away from both L_∞ (too large) and $1/L_\infty$ (too small) by some finite power of 2.

Stability of $\mathbb{R}$ under the operations

Stability properties

- $X, Y \in \mathbb{R} \Longrightarrow X + Y \in \mathbb{R}$.
- $X, Y \in \mathbb{R} \Longrightarrow X \cdot Y \in \mathbb{R}$.
- $X \in \mathbb{R} \Longrightarrow -X \in \mathbb{R}$.
- $X \in \mathbb{R}$, $X \neq 0 \Longrightarrow X^{-1} \in \mathbb{R}$.

Key arguments.

- If $|X|, |Y| < 2^m$, then $|X + Y| < 2^{m+1}$ and $|X \cdot Y| < 2^{2m}$, still standard-bounded.
- If $|X| > 2^{-m}$, then $|X^{-1}| < 2^m$, still standard.

Subtlety. Non-infinitesimality is not automatic under subtraction in $\widetilde{\mathbb{R}}$. Maintaining $\mathbb{R}$ closed requires interpreting the definition as a topological closure (rigorous formulation deferred).

$\mathbb{R}$ is a (total) field

Theorem (Field structure)

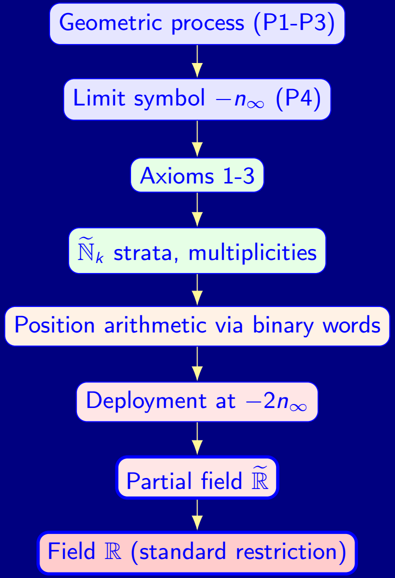
$(\mathbb{R}, +, \cdot)$ is a commutative field in the classical sense:

1. $(\mathbb{R}, +)$ is an abelian group (total addition).
2. $(\mathbb{R} \setminus \{0\}, \cdot)$ is an abelian group (total multiplication; every non-zero element invertible).
3. Distributivity.

Crucial point. On $\mathbb{R}$, the partial operations of $\widetilde{\mathbb{R}}$ become *total*: stability ensures that results never escape the standard zone.

$\widetilde{\mathbb{R}}$ is naturally a non-archimedean extension of $\mathbb{R}$: it contains infinitely large elements (such as 2^{n_∞}) and non-zero infinitesimals (such as $1/2^{n_\infty}$).

The construction in one picture



What this volume accomplishes

1. **Geometric foundation.** The natural numbers, the continuum, and the real field arise from a single geometric mechanism: circles with angle doubling.
2. **Stratified multiplicities.** A precise hierarchy $\widetilde{\mathbb{N}}_0 \subset \widetilde{\mathbb{N}}_1 \subset \widetilde{\mathbb{N}}_2 \subset \cdots$ based on three minimal axioms.
3. **Position arithmetic.** Multiplication of positions on C_1 rigorously derived from binary lexical multiplication: not posited *ad hoc*.
4. **Partial field $\widetilde{\mathbb{R}}$.** A non-archimedean extension with explicit invertibility thresholds $1/L_\infty$ and L_∞.
5. **Classical field $\mathbb{R}$.** Recovered as the standard restriction, with total operations after the partial ones are confined to the standard zone.

Outlook

- **Volume II — Resolution symmetry.** The geometric process is bidirectional: refinement ($\mathcal{F}$) and dilation ($\mathcal{F}^{-1}$) are dual. The line at infinity Δ_∞ carries a double mosaic structure symmetric to the deployment $\widetilde{\mathbb{R}}^+$.
- **Volume III — Divergent integrals.** Applying the stratified framework to divergent integrals: the regularization stratum $\sigma(f)$, the critical bound $B_k(f)$, and universal regularization $\mathrm{Reg}_k(f) = 1$.
- **Volume IV (forthcoming) — Algebraic auto-duality.** The involution $\iota : X \mapsto 1/X$ on $\widetilde{\mathbb{R}}^*$ and its relation to the geometric duality δ_k. Subject for further work.
- **Open questions. Topological properties** on $\widetilde{\mathbb{R}}$, non-archimedean analysis on $\widetilde{\mathbb{R}}$; potential applications to regularization in physics.

Thank you.

Questions?
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May 25, 2026