The Absolute Point Is a Grain

From Existence to Construction

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Outline

1. That the absolute point is a grain
2. Transition: the self-similarity of the grain
3. The constructed grain

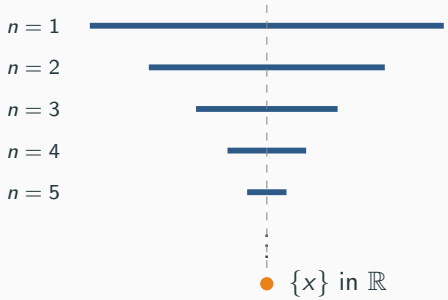
Thesis

What we take for a point is a *grain* seen from too far away. We show it twice: first through **existence** (cardinality, Hölder), then through **construction** (non-idempotence, Peano).

That the absolute point is a grain

Euclid or Aristotle?

- **Euclid**: "a point is that which has no part."
- **Aristotle**: the line is endlessly divisible; it is not made of points.
- The real line sides with Euclid: endless division ends in a point of zero thickness.



The absolute point

Definition 1

In an ordered field $\mathbb{K}$, an element x is an absolute point if

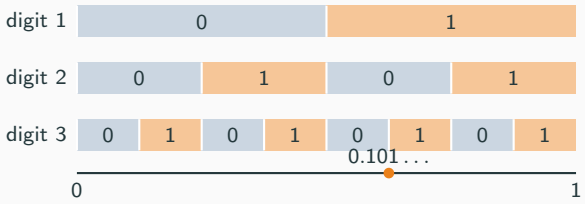
$$\bigcap_{n\geq 1} \left[x - \tfrac{1}{n},\ x + \tfrac{1}{n}\right] = \{x\}.$$

- In $\mathbb{R}$: **every** point is absolute.
- Question: is this a necessity of the continuum, or a consequence of its *size*?

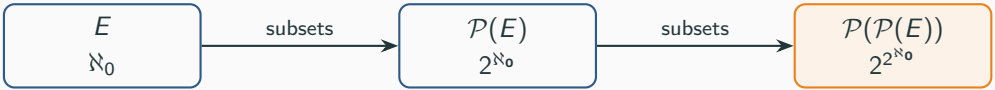


First tier: $\mathcal{P}(\mathbb{N})$ indexes the segment

- $A \subset \mathbb{N} \;\mapsto\; f(A) = \sum_{n \in A} 2^{-(n+1)}$.
- A subset = a sequence of left/right choices = a point.
- After identifying the dyadics: bijection $\mathcal{P}(\mathbb{N}) \simeq [0, 1]$.
- $\aleph_0$ positions, $2^{\aleph_0}$ descriptions.



Second tier: $\mathcal{P}(\mathcal{P}(E))$



indexes $[0,1] \subset \mathbb{R}$
absolute points

indexes a continuum $\mathbb{K}$
finer?

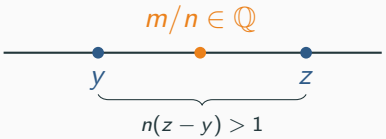
- By Cantor, $2^{2^{\aleph_0}} > 2^{\aleph_0}$: $\mathbb{K}$ cannot be the real segment.
- If $\mathbb{K}$ is an ordered field containing $\mathbb{R}$, what becomes of its points?

Hölder's lemma

Theorem 2 (Hölder, 1901)

*An **Archimedean** ordered field embeds into $\mathbb{R}$ in an order-preserving way. It therefore has at most $2^{\aleph_0}$ elements.*

Idea. Archimedes $\Rightarrow$ between $y < z$ there is a rational m/n.
Hence $\varphi(y) = \sup\{q \in \mathbb{Q} : q < y\}$ is strictly increasing.



Consequence

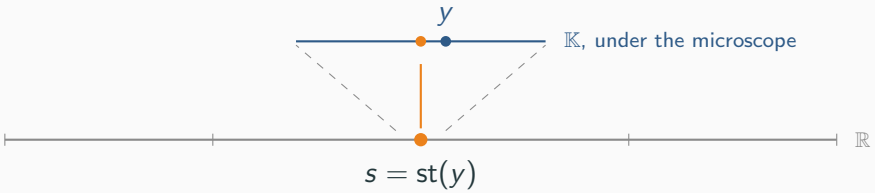
An ordered field with $2^{2^{\aleph_0}}$ points is **non-Archimedean**: it contains an $H > n$ for every integer n, hence an infinitesimal $\varepsilon = 1/H$.

The shadow lemma

Theorem 3 (Shadow)

Every limited element $y \in \mathbb{K}$ is infinitely close to a unique real, its shadow $\mathrm{st}(y)$.

Proof. $s = \sup\{r \in \mathbb{R} : r < y\}$; then $|y - s| \leq 1/n$ for every n.



Theorem 4 (The absolute point is a grain)

Let E be countable and $\mathbb{K} \supset \mathbb{R}$ an ordered field bijectively indexed by $\mathcal{P}(\mathcal{P}(E))$. Then every real x is a grain relative to $\mathbb{K}$: its monad

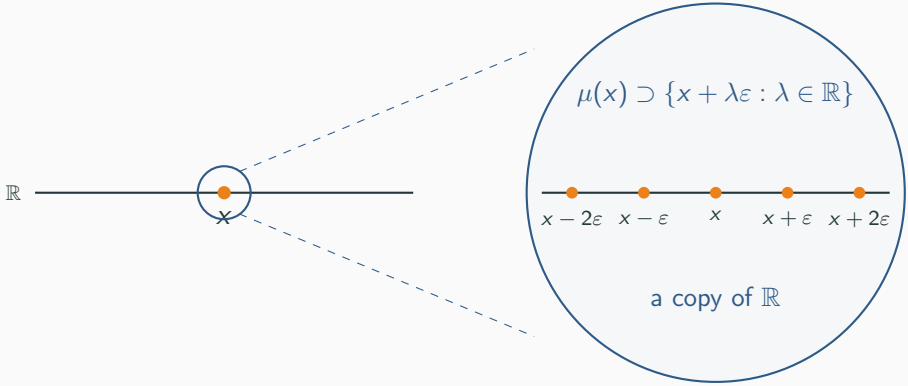
$$\mu(x) = \{y \in \mathbb{K} : y - x \text{ infinitesimal}\}$$

contains a copy of $\mathbb{R}$.

Proof.

1. $|\mathbb{K}| = 2^{2^{\aleph_0}} > 2^{\aleph_0}$, so $\mathbb{K}$ is non-Archimedean (Hölder): $\exists\, \varepsilon > 0$ infinitesimal.
2. $\lambda \mapsto x + \lambda\varepsilon$ embeds $\mathbb{R}$ into $\mu(x)$.
3. $\bigcap_n [x - \frac{1}{n}, x + \frac{1}{n}] = \mu(x) \neq \{x\}$: no point of $\mathbb{K}$ is absolute.

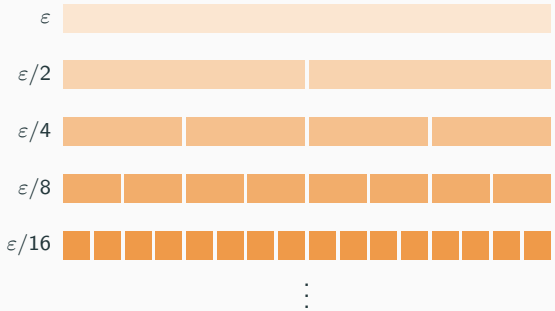
Under the microscope: the point becomes a continuum



What we see as the point x is the shadow of a grain that has the cardinality of the continuum.

No minimal grain

- $0 < \varepsilon/2 < \varepsilon < 1/n$ for every n.
- Each grain splits into two grains that are still infinitesimal.
- The division goes on without end: this is Aristotle's line.



Cardinality and thickness

Tier	Indices	Cardinality	What a point is
first, $[0,1] \subset \mathbb{R}$	$\mathcal{P}(E)$	$2^{\aleph_0}$	absolute point, zero thickness
second, $\mathbb{K}$	$\mathcal{P}(\mathcal{P}(E))$	$2^{2^{\aleph_0}}$	grain: $\mathrm{st}^{-1}(0_{\mathbb{R}}) = \mu(0)$

- **Thickness is a number, not a cardinal**: since $2 \cdot 2^{\aleph_0} = 2^{\aleph_0}$, a grain "$1/2^{\aleph_0}$" would be indivisible.
- **Cardinality forces grains without fixing their thickness**: ε, $\varepsilon/2$, $\varepsilon^2, \ldots$
- **Existence**: the ultrapower $\mathbb{R}^{\mathbb{R}}/\mathcal{U}$ has $2^{2^{\aleph_0}}$ elements (Frayne–Morel–Scott).

Limit of this approach

It proves that the grain *exists*, by contradiction, without *showing* it.

Application: the Dirac distribution

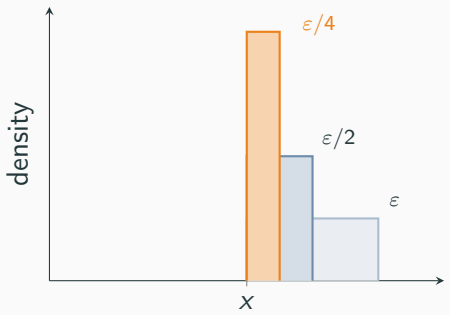
Density on the grain $[x, x+\varepsilon]$:

$$\delta_\varepsilon = \tfrac{1}{\varepsilon} \text{ on the grain}, \quad \tfrac{1}{\varepsilon} \cdot \varepsilon = 1.$$

For polynomial f:

$$\tfrac{1}{\varepsilon} \int_x^{x+\varepsilon} f = f(x) + \tfrac{\varepsilon}{2} f'(x) + \cdots$$

$$\operatorname{st}\Big(\int f \, \delta_\varepsilon \Big) = f(x) = \langle \delta_x, f \rangle.$$



Transition: the self-similarity of the grain

Each grain is a copy of the segment

At level ν, $\#[0,1[= 2^\nu$ grains:

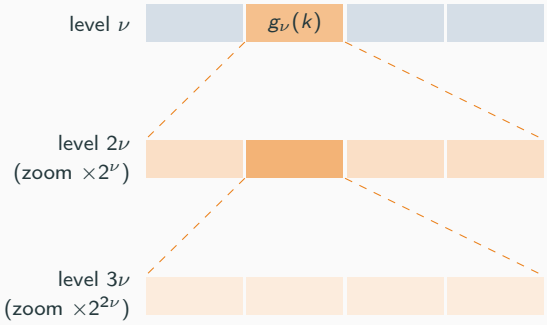
$$g_\nu(k) = \frac{k}{2^\nu} + 2^{-\nu}\,[0,1[.$$

Each grain is the whole segment, scaled down by a factor $2^{-\nu}$.

Observing each grain at level ν: $\#[0,1[= 2^\nu \cdot 2^\nu = 2^{2\nu}$.

Repeating inside each grain: $2^{2\nu} \cdot 2^\nu = 2^{3\nu}$, and so on.

Each step adds ν.



the same pattern at every scale

Self-similarity counts; cardinality does not

Theorem 5 (Self-similarity of the grid: a step of ν)

$$\mathbb{G}_{(j+1)\nu} = \bigcup_{x \in \mathbb{G}_{j\nu}} \left(x + 2^{-j\nu}\, \mathbb{G}_\nu \right), \qquad \textit{disjoint union.}$$

Hence $\#\mathbb{G}_{(j+1)\nu} = 2^{j\nu} \cdot 2^\nu$*:* $\nu \to 2\nu \to 3\nu \to \cdots$

Proof. $\dfrac{n}{2^{(j+1)\nu}}$ with $n = k\,2^\nu + i$, $0 \le i < 2^\nu$ (Euclidean division) $= \dfrac{k}{2^{j\nu}} + 2^{-j\nu}\dfrac{i}{2^\nu}$.

With Cantor

$2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0}$:

the copy of itself inside each point *does not show* in the count.

In the grain

$2^\nu \cdot 2^\nu = 2^{2\nu} \neq 2^\nu$:

self-similarity *produces* a strictly finer level.

The constructed grain

Two roads to the grain

	Road through cardinality	Road through non-idempotence
Resource	$2^{2^{\aleph_0}}$, Cantor, Hölder	an infinite integer ν
Framework	set theory, choice	Peano arithmetic
Nature	existence, by contradiction	explicit construction
Result	$\mu(x)$ contains a copy of $\mathbb{R}$	$\Gamma(x)$ contains 2^{ν} points

The first shows that the grain is **unavoidable**; the second shows **how it is made**.

The framework: a nonstandard model of Peano arithmetic

- **Skolem (1934)**: there is a model $\mathcal{M}$ satisfying the same arithmetical sentences as $\mathbb{N}$, but containing infinite integers.
- **Transfer**: an arithmetical sentence true in $\mathbb{N}$ is true in $\mathcal{M}$.
- We fix an infinite $\nu = \#\mathbb{N}$.

Non-idempotence

$x + x = x \Rightarrow x = 0$, so $2\nu \neq \nu$;
$2^{2\nu} = 2^\nu \cdot 2^\nu > 2^\nu$.

Infinitesimals

m infinite $\Rightarrow 2^{-m} < 1/n$ for every standard n
(since $2^m > m^2$ for $m \geq 5$).

Neither Cantor's set theory nor the axiom of choice.

Grids, grains and lower measure

$$\mathbb{G}_m = \left\{ \tfrac{k}{2^m} : 0 \le k < 2^m \right\} \subset [0,1[$$

- 2^m points, **exact** binary expansion with m digits.
- Grain: $g_m(k) = \left[\tfrac{k}{2^m}, \tfrac{k+1}{2^m}\right[$.
- Lower measure: $m_{\text{inf}} = \tfrac{k}{2^m}$ (left endpoint).
- Tiling: $1 = 2^m \cdot 2^{-m}$, i.e. $L = |L| \cdot \varepsilon$.

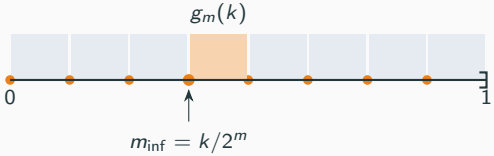


diagram: a few grains stand for the 2^m

Embedding by zeros

$$\iota : \mathbb{G}_\nu \to \mathbb{G}_{2\nu}, \qquad 0.\varepsilon_1 \dots \varepsilon_\nu \longmapsto 0.\varepsilon_1 \dots \varepsilon_\nu \underbrace{0 \dots 0}_{\nu}$$

- Same value: $\dfrac{k}{2^\nu} = \dfrac{k\,2^\nu}{2^{2\nu}}$. Injective, order-preserving.
- Image = the points of $\mathbb{G}_{2\nu}$ whose last ν digits are zero.
- $2^{2\nu} - 2^\nu$ new points appear.

Analogy

Level ν is to level 2ν what $\mathbb{Z}$ is to $\mathbb{R}$: a point of level 2ν is "integral" if it has only zeros after place ν.

Theorem: the point is a grain (constructive version)

Theorem 6

For $x \in \mathbb{G}_\nu$, let $\Gamma(x) = \mathbb{G}_{2\nu} \cap [x,\, x + 2^{-\nu}[$. Then:

1. $\Gamma(x) = \{x + j\,2^{-2\nu} : 0 \le j < 2^\nu\}$ *has exactly* 2^ν *points;*
2. x *is its lower measure;*
3. *all its points are infinitely close to* x*;*
4. *the sets* $\Gamma(x)$ *form a partition of* $\mathbb{G}_{2\nu}$*;*
5. *length is preserved:* $2^\nu \cdot 2^\nu \cdot 2^{-2\nu} = 1$.

Proof of 3. $0 \le j\,2^{-2\nu} < 2^{-\nu}$, which is infinitesimal since ν is infinite.

The grain, pictured

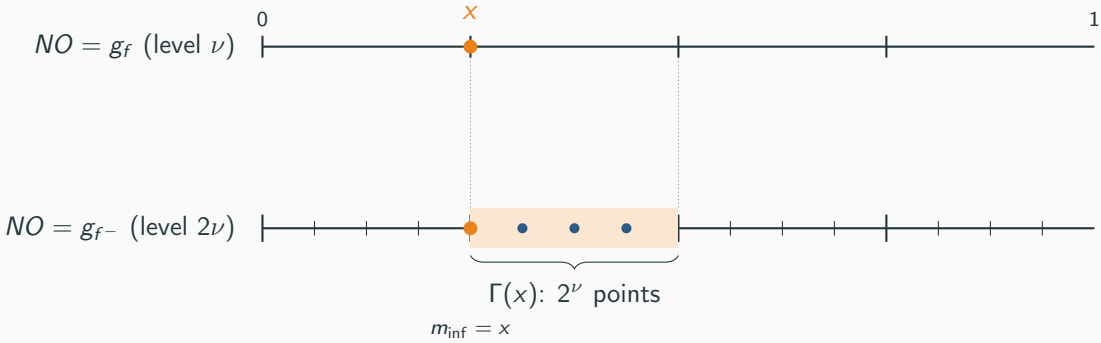


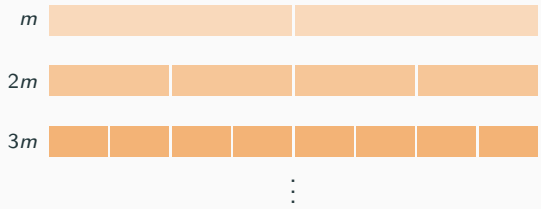
Diagram: the $\#\mathbb{N}$ positions are represented by a few of them. With $\nu = \#\mathbb{N}$, the 2^ν points of $\Gamma(x)$ are indistinguishable at the standard scale: the standard observer takes them for a single point.

No absolute point, at any level

Theorem 7 (Corollary)

For every level $m \geq 1$ and every $x \in \mathbb{G}_m$, the grain of x at level $2m$ contains $2^m \geq 2$ points.

- True at every level, finite or infinite.
- What infinity adds: the **indistinguishability** of the points of the grain.

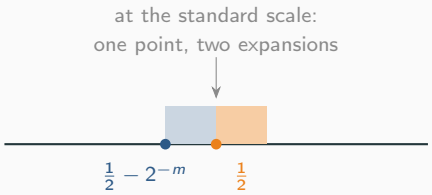


Expansions are unique

At a fixed level m, a single expansion:

$$0.0\underbrace{1\ldots1}_{m-1} = \tfrac{1}{2} - 2^{-m} \neq 0.1\underbrace{0\ldots0}_{m-1} = \tfrac{1}{2}$$

- These are two **neighboring** grains.
- An exact description, with no convergence.



Trace

The double expansion $0.1000\ldots = 0.0111\ldots$ in $\mathbb{R}$ is the **trace** of two neighboring grains conflated.

The hierarchy of levels

$$\mathbb{G}_\nu \subset \mathbb{G}_{2\nu} \subset \mathbb{G}_{3\nu} \subset \cdots \qquad 2^\nu < 2^{2\nu} < 2^{3\nu} < \cdots$$

Level	Notation	Points	Thickness	Digits
ν	$NO = g_f$	2^ν	$2^{-\nu}$	ν
2ν	$NO = g_{f-}$	$2^{2\nu}$	$2^{-2\nu}$	2ν
3ν	$NO = g_{f--}$	$2^{3\nu}$	$2^{-3\nu}$	3ν

Two steps not to be confused. Self-similarity of ratio 2^ν adds ν: $2\nu \to 3\nu$. The product *adds* levels: two factors of level 2ν give 4ν, skipping 3ν.

What Cantor cannot do

With $2^{2\aleph_0} = 2^{\aleph_0}$, the levels would merge. Here each one is strictly finer than the previous one.

Operations, and the grain of $\frac{1}{3}$

- **Addition**: $x, y \in \mathbb{G}_m \Rightarrow x + y \in \mathbb{G}_m$ (if < 1).
- **Product**: $xy = \frac{kk'}{2^{2m}} \in \mathbb{G}_{2m}$: we go up one level.
- **Grains carried by the reals**: each absolute point x becomes

$$g_\nu(x) = [x,\ x + 2^{-\nu}[, \qquad m_{\inf} = x.$$

Two distinct reals have disjoint grains.

$\frac{1}{3}$ is the **lower measure of its grain**, just as it is the measure of its point. Its grain is not a grid grain: it straddles two of them.

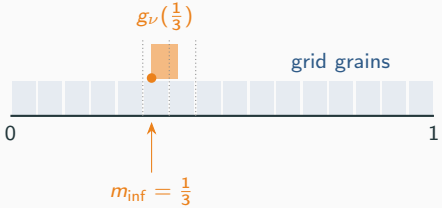


diagram: a few grains for $2^{\#\mathbb{N}}$

Division goes down one level: the halo of $\frac{1}{3}$

Long division

$\nu \times \nu$ digits $\to 2\nu$ digits, zeros included: multiplication goes up one level; division goes down one.

$$q_{2\nu} = \left(\frac{1_{2\nu}}{3_\nu}\right)_{2\nu} = 0.0101\ldots, \qquad 1_{2\nu} = 1.\underbrace{0\ldots0}_{2\#\mathbb{N}}.$$

$$1 = 3\,q_{2\nu} + r \cdot 2^{-2\nu}, \quad r \in \{1, 2\}.$$

$r \neq 0$: $3x = x + 2x$ keeps the last nonzero digit of x.

The halo

$$0 < \tfrac{1}{3} - q_{2\nu} < 2^{-2\nu}.$$

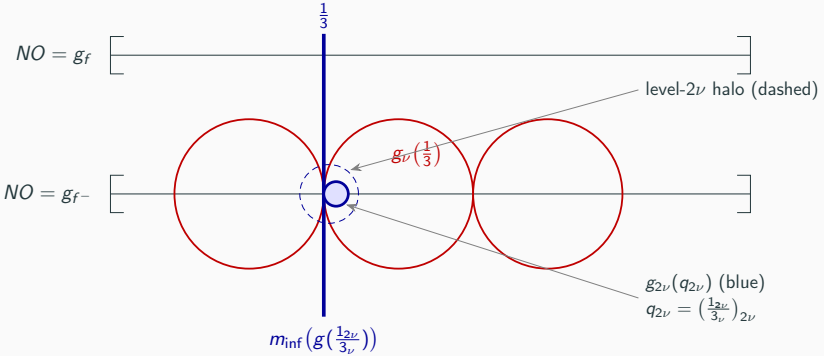
$m_{\inf}\big(g_\nu(\tfrac{1}{3})\big) = \tfrac{1}{3}$ lies in the halo of $q_{2\nu}$ at level $\mathbb{G}_{2\nu}$.

At level ν, $q_{2\nu}$ and $\frac{1}{3}$ are **the same point**: the gap is 2^ν times smaller than a grain.

$\frac{1}{3}$: designated, computed, and its halo

Designated at level ν as the lower measure of its grain; computed from level 2ν; the remainder lodges in its halo. In $\mathbb{R}$, it vanishes: $3 \times 0.0101\ldots = 0.1111\ldots = 1$.

The halo of $\frac{1}{3}$, pictured



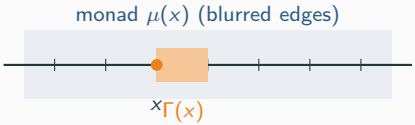
In red, the level-ν grains carried by the reals; $g_\nu(\frac{1}{3})$ starts at $\frac{1}{3}$. In blue, the level-2ν grain of the computed quotient; dashed, its halo, which contains $\frac{1}{3}$. At level g_f, the two lower measures coincide.

Grain and monad

Theorem 8

For $x \in \mathbb{G}_\nu$*:* $\quad \Gamma(x) \subsetneq \mu_{2\nu}(x) = \{y \in \mathbb{G}_{2\nu} : y \simeq x\}$.

	Grain	Monad
shape	$[x, x + 2^{-\nu}[$	$y \simeq x$
points	exactly 2^ν	undefined
position of x	left endpoint	interior
neighbors	disjoint	merged



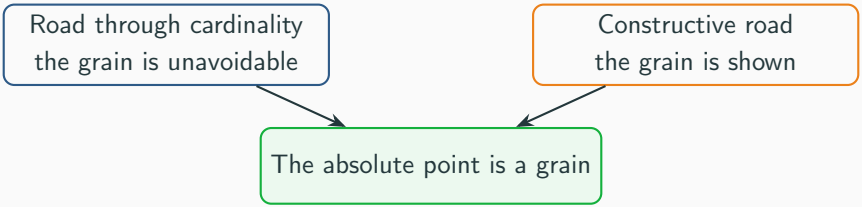
The monad has no number of points in $\mathcal{M}$: "infinitely close" is not an arithmetical formula — which is also why the sorites paradox does not apply.

The two roads meet

- The fractions of $\mathcal{M}$ form a **non-Archimedean ordered field**, which contains the whole hierarchy of grids.
- If $\mathcal{M}$ is countable (Skolem allows it), so is this field.

Consequence

The grain does not require $2^{2^{\aleph_0}}$ points. Cardinality is a *sufficient* condition; non-idempotence shows that the grain is already there as soon as one has an infinite integer.



Conclusion

Conclusion

1. **Existence.** An ordered field continuum indexed by $\mathcal{P}(\mathcal{P}(E))$ is non-Archimedean: each real is the shadow of a monad containing a copy of $\mathbb{R}$.
2. **Self-similarity.** Each grain is a scaled-down copy of the segment: $2^\nu \cdot 2^\nu = 2^{2\nu} \neq 2^\nu$; each step adds ν.
3. **Construction.** At level 2ν, each point of level ν becomes the lower measure of a grain of 2^ν infinitely close points.
4. **Division.** $\frac{1}{3}$ is the lower measure of its grain: designated at level ν, computed from level 2ν, the remainder lodges in its halo.

Length is preserved, grains divide, and division never stops:
the absolute point is a grain seen from too far away.

To close: Dirac's original delta

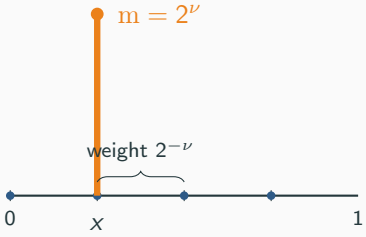
Dirac's original delta: a density at a single point

Integral at level m:

$$\int_{\mathbb{G}_m} f = \sum_{y \in \mathbb{G}_m} f(y)\, 2^{-m}.$$

Dirac (1930): a density m at one point and one point only,

$$\delta_x = \mathrm{m} \cdot 1_{\{x\}}, \qquad x \in \mathbb{G}_\nu.$$



level ν (diagram)

- Mass: $\int \delta_x = \mathrm{m} \cdot 2^{-\nu} = 1 \iff \mathrm{m} = 2^\nu$.
- Action: $\int f\, \delta_x = f(x) \cdot 2^\nu \cdot 2^{-\nu} = f(x)$, exactly.

In $\mathbb{R}$ a point has measure zero; at level ν it carries a weight $2^{-\nu}$: Dirac's original delta becomes **exact**.

Passing to $\mathbb{G}_{2\nu}$: the point becomes a neighborhood

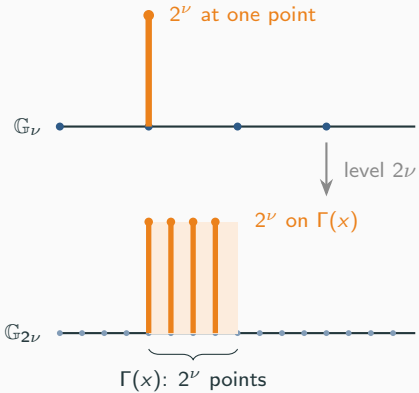
At level 2ν, the point density is read on the grain of x:

$$\delta_x = 2^\nu \cdot 1_{\Gamma(x)} \quad \text{on } \mathbb{G}_{2\nu},$$

where $\Gamma(x) = [x,\, x + 2^{-\nu}[\cap \mathbb{G}_{2\nu}$ is the neighborhood of x in $\mathbb{G}_\nu$, its grain.

- Mass: $2^\nu \cdot 2^\nu \cdot 2^{-2\nu} = 1$.
- Action: the **average** of f over the grain,

$$\int f\, \delta_x = \tfrac{1}{2^\nu} \sum_{j<2^\nu} f\big(x + j\, 2^{-2\nu}\big) \simeq f(x).$$



The equivalence of the two Dirac deltas

Theorem 9

For $x \in \mathbb{G}_\nu$ and f polynomial with standard coefficients:

$$\underbrace{\int_{\mathbb{G}_\nu} f \cdot 2^\nu 1_{\{x\}}}_{\text{point Dirac, level } \nu} = f(x) \simeq \underbrace{\int_{\mathbb{G}_{2\nu}} f \cdot \tfrac{1}{\varepsilon} 1_{[x,x+\varepsilon[}}_{\text{Dirac on a neighborhood, level } 2\nu}, \qquad \varepsilon = 2^{-\nu}.$$

Proof. On $[0,1]$, $|f(x+t) - f(x)| \leq C\,t$ with C standard; for $t < 2^{-\nu}$, the gap is $< C\,2^{-\nu}$, which is infinitesimal.

The point is a grain

What is a density **at a point** at level ν is a density **on a neighborhood** at level 2ν. Dirac's original delta and Schwartz's delta are the same object, seen at two levels of observation.

Thank you

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