Towards the Complex Plane

From the exponential scale to the anti-Dirac

Hugues Genvrin

November 14th, 2026

Outline

1. Framework
2. The exponential scale of dilations
3. The exponential scale on the circle
4. Towards i
5. From the circle to the complex plane
6. From the anti-Dirac to the complex plane
7. Synthesis

Framework

Grains and observation levels

We work in a nonstandard model of arithmetic, in Skolem's sense, with a fixed unlimited integer n_∞.

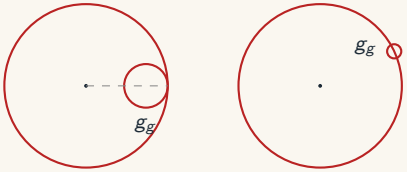
Definition

A grain is an extent taken as the unit at some observation level NO. We consider four nested grains, each one 2^{n_∞} times the previous one:

$$g_{f-} < g_f < g_g < g_{g+}, \qquad g_f = 2^{n_\infty} g_{f-}, \quad g_g = 2^{n_\infty} g_f, \quad g_{g+} = 2^{n_\infty} g_g.$$

At the level $NO = g$, the grain g is a point: its parts cannot be told apart.

Two situs, two readings



Radial situs Circular situs

$NO = g_f$

A grain g_g is carried either by a radius or by the circumference.

The grain $2^k g_g$ reads in two equivalent ways:

- the image of g_g under a dilation of ratio 2^k;
- a grain that contains more grains g_f.

Proposition

$2^k\, g_g = 2^{n_\infty + k}\, g_f$, and in particular $2^{n_\infty}\, g_g = 2^{2n_\infty}\, g_f$.

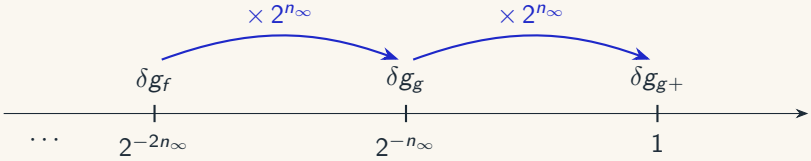
The exponential scale of dilations

Thickness

With $NO = g_g$, the grain $g_{g+} = 2^{n_\infty}\, g_g$ is the dilation of a point. The thickness δg of a grain is its measure when g_{g+} is the unit.

Proposition

$\delta g_g = 2^{-n_\infty}$, $\delta g_f = 2^{-2n_\infty}$, *and* $\delta g_{g+} = 2^{n_\infty} \cdot 2^{-n_\infty} = 1$.



An exponential law, with a limitation

Composing two dilations adds their exponents:

$$2^k \cdot 2^{k'} \;=\; 2^{k+k'}.$$

What this scale cannot do

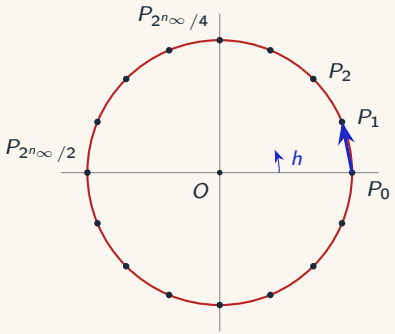
It takes only *positive* values and it is strictly monotone: a repeated dilation never comes back to its starting point.

No element of this scale has a negative square.

The minus sign of the complex plane has to come from somewhere else.

The exponential scale on the circle

Equally spaced points



On the unit circle we place 2^{n_∞} equally spaced points

$$P_0,\ P_1,\ \ldots,\ P_{2^{n_\infty}-1},$$

starting from the tip P_0 of the horizontal radius.

Angular step: $h = \dfrac{2\pi}{2^{n_\infty}}$.

The point P_m has curvilinear abscissa $\mu = m\,h$.

(16 points are drawn instead of 2^{n_∞}.)

The product of points of the circle

We transport the law of dilations to the circle: the product of two points is the point whose curvilinear abscissa is the sum of the abscissas.

Definition

$P_a \cdot P_b = P_{a+b}$, the index being read modulo 2^{n_∞}.

Proposition

The points of the grid form a commutative group with 2^{n_∞} elements and identity P_0. Multiplying by P_a is the rotation about O taking P_0 to P_a.

Same law, but the scale is now periodic: $P_{2^{n_\infty}} = P_0$. This is what makes a negative square possible.

The exponential notation

Lemma (The first step is vertical)

Let u be the vertical unit vector. Then $\overrightarrow{P_0P_1} = h\,u + \varepsilon$, with ε infinitely small compared with h.

Indeed $\overrightarrow{P_0P_1} = (\cos h - 1,\ \sin h)$, with $|\cos h - 1| \leq h^2/2$ and $|\sin h - h| \leq h^3/6$.

The arc leaving P_0 advances by vertical grains: $\mu = \theta \cdot g_g^{\downarrow}$.

Definition

For $\theta = m\,h$, we write $e^{\theta\, g_g^{\downarrow}} = P_m$.

The notation is justified by $e^{\theta g_g^{\downarrow}} \cdot e^{\theta' g_g^{\downarrow}} = e^{(\theta+\theta')g_g^{\downarrow}}$ and $e^{hg_g^{\downarrow}} - e^0 = h\,g_g^{\downarrow} + \varepsilon$.

Towards $\imath$

Two rays, one convention

Two circles are superposed: g_g° reads as 2^{n_∞} grains g_f°, or as the dilation of g_f°. Relative to the pair of levels $NO = (g_f, g_{f-})$ we consider the horizontal ray g_f^{-} and the vertical ray $g_g^{|}$.

Convention (normalised reading)

Each ray is the unit of its own axis. On the unit circle,

$$g_f^{-} = P_0 = 1, \qquad g_g^{|} = P_{2^{n_\infty}/4} = e^{\frac{\pi}{2} g_g^{|}}.$$

The grain does the work

$P_{2^{n_\infty}/4}$ exists because 2^{n_∞} is divisible by 4: the quarter turn falls *exactly* on a point of the grid, with no rounding.

The square of the quarter turn

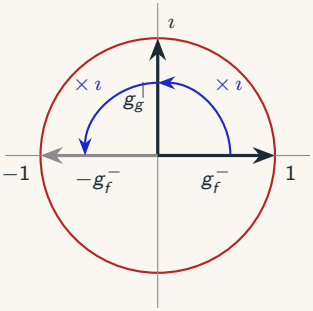
Theorem

$$\left(g_g^{\lfloor}\right)^2 = \left(e^{\frac{\pi}{2} g_g^{\lfloor}}\right)^2 = e^{\pi\, g_g^{\lfloor}} = -\,g_f^{-}.$$

On the unit circle $g_f^{-} = 1$, *hence*

$$\left(g_g^{\lfloor}\right)^2 = -1, \qquad g_g^{\lfloor} = \imath.$$

Proof. $P_{2^{n_\infty}/4} \cdot P_{2^{n_\infty}/4} = P_{2^{n_\infty}/2}$: two quarter turns make a half turn, and $P_{2^{n_\infty}/2}$ is diametrically opposite to P_0. □



Consistency and change of level

Corollary (Consistency with the reals)

$(-1)^2 = 1, \quad \imath^3 = -\imath, \quad \imath^4 = 1.$

$P_{2^{n_\infty}/2} \cdot P_{2^{n_\infty}/2} = P_{2^{n_\infty}} = P_0$, and similarly for the other two.

Remark (change of level)

In the normalised reading, $\imath$ sends the unit of one axis to the unit of the other.

Measured with the same fine unit, the vertical ray belongs to the coarse level, and the change of level 2^{n_∞} must be added:

$$\omega = 2^{n_\infty}\,\imath, \qquad \omega^2 = -2^{2n_\infty}.$$

From the circle to the complex plane

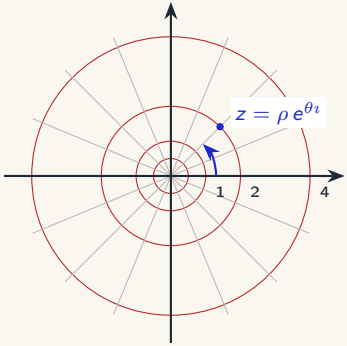
Combining the two scales

The circle carries the rotations; the scale of thicknesses carries the dilations.

Definition

For $z \neq O$, let M_z be the direct similarity about O taking $g_f^- = 1$ to z: a dilation of ratio ρ composed with a rotation of angle θ. We write

$$z = \rho\, e^{\theta\, \imath}, \qquad z \cdot w = M_z(w).$$



The complex plane

Theorem

1. *The product extends that of the circle and that of the dilations:* $\rho\, e^{\theta \imath} \cdot \rho'\, e^{\theta' \imath} = \rho\rho'\, e^{(\theta+\theta')\imath}$.
2. *Every point is written uniquely* $z = x + y\,\imath$, *with* x, y *real.*
3. *The product is commutative, distributive over vector addition, and*

$$(x + y\,\imath)(x' + y'\,\imath) = (xx' - yy') + (xy' + x'y)\,\imath.$$

The plane, with vector addition and this product, is the field of complex numbers.

Key step. In the basis $(1, \imath)$, $M_z = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$: linear in w and additive in z.

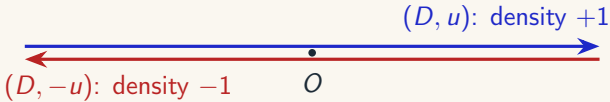
From the anti-Dirac to the complex plane

The anti-Dirac

An oriented line (D, u) carries the oriented measure $m_u(\overrightarrow{pq}) = t_q - t_p$, where $p = t_p\, u$, $q = t_q\, u$.

Definition

The anti-Dirac carried by D is the superposition of (D, u) and $(D, -u)$. Its total measure vanishes on every segment: $m_u + m_{-u} = 0$.



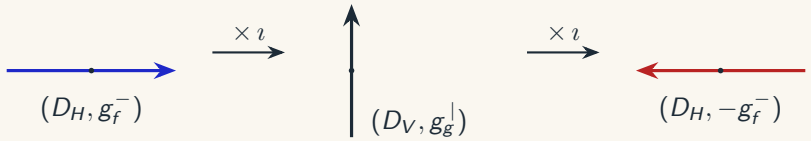
The Dirac concentrates a mass 1 in a single grain; the anti-Dirac spreads two opposite densities over the whole line. It is *equivalent* to zero without being equal to it: a zero of equivalence.

Theorem

Let $H = (D_H, g_f^-)$ and $V = (D_V, g_g^\downarrow)$ be the oriented axes. Multiplying by $g_g^\downarrow = \imath$ sends H to V, then V to $(D_H, -g_f^-)$. Hence

$$H \quad \text{and} \quad \left(g_g^\downarrow\right)^2 \cdot H = (D_H, -g_f^-)$$

are two superposed lines with opposite oriented measures: an anti-Dirac.



At the scale of the grains

Let $g^{|}_{g+} = 2^{n_\infty}\, g^{|}_{g}$ be the vertical ray of the next level and $g^{-}_{g} = 2^{n_\infty}\, g^{-}_{f}$ the coarse horizontal ray.

Corollary

$$g^{|}_{g} \cdot g^{|}_{g+} \;=\; 2^{n_\infty} \left(g^{|}_{g}\right)^2 \;=\; -\,2^{n_\infty}\, g^{-}_{f} \;=\; -\,g^{-}_{g}.$$

The product of the two vertical rays of adjacent levels is the opposite of the coarse horizontal ray.

The lines carried by g^{-}_{g} and $-g^{-}_{g}$ are superposed: they form an anti-Dirac at the coarse level.

The converse: a square root of the half turn

Start from an anti-Dirac (D, u), $(D, -u)$. Going from one line to the other is the half turn -1.

Building the complex plane means finding a square root of this half turn.

Lemma

There is no square root of the half turn inside the line D.

The only isometries of D fixing O are the identity and the half turn, and both square to the identity.

We must leave the line by doubling it: a second line D' through O, perpendicular to D, and an exchange J, that is, an isometry fixing O which swaps D and D'.

Four exchanges, two square roots

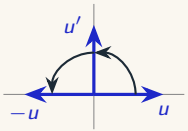
Theorem

There are exactly four exchanges

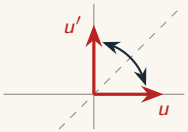
$$J_{a,b}:\ u \mapsto a\,u', \quad u' \mapsto b\,u, \qquad a, b = \pm 1,$$

and $J_{a,b}^2 = ab$.

- *Direct* ($ab = -1$)*: a quarter turn,* $J^2 = -1$*, and the plane with* J *is the complex plane,* $J = \pm\imath$.
- *Indirect* ($ab = +1$)*: a reflection,* $J^2 = +1$*; no complex plane.*



direct: $J^2 = -1$



indirect: $J^2 = +1$

Synthesis

Equivalence

Theorem

The following two data are equivalent:

1. *the complex plane, that is, the plane with a quarter turn $\imath$;*
2. *an anti-Dirac carried by a line D, a line D' perpendicular to D, and a direct exchange between D and D'.*

Scope of the converse

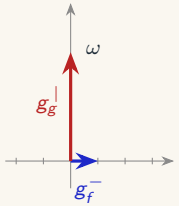
The anti-Dirac alone is not enough: it supplies the half turn, not its square root. The extra data are the doubling of the line and the direct character of the exchange. The construction is then unique up to conjugation.

The plane with two adjacent observation levels

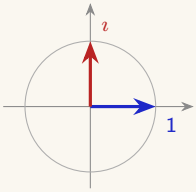
Definition

The plane Π carries the horizontal ray g_f^-, read at the fine level, and the vertical ray $g_g^|$, read at the coarse level:

$$p = x\, g_f^- + y\, g_g^|.$$



common: $\omega^2 = -2^{2n_\infty}$



normalised: $\imath^2 = -1$

(ratio 2^{n_∞} drawn as 4)

Two readings:

- normalised: $\nu(p) = x + y\,\imath$;
- common: $\kappa(p) = x + y\,\omega$, with $\omega = 2^{n_\infty}\,\imath$.

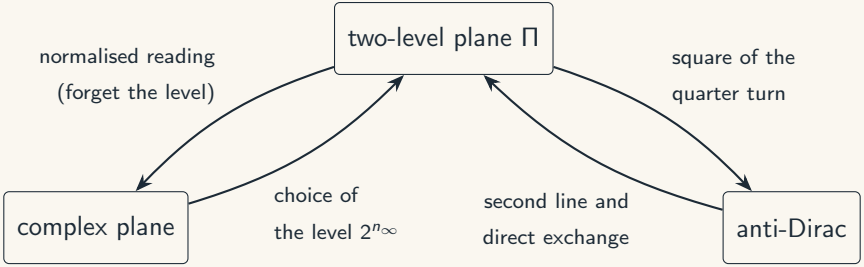
Two-level plane, complex plane and anti-Dirac

Theorem

Let Π be a plane with two adjacent observation levels.

1. *Equivalence. The normalised reading ν is a bijection from Π onto the complex plane preserving sum and product.*
2. *No equality. The two readings differ by the affinity $A : x + y\,\imath \mapsto x + 2^{n_\infty}\, y\,\imath$, which preserves the sum but is neither an isometry nor compatible with the product.*
3. *Π carries an anti-Dirac, given at the coarse level by $g_g^{\downarrow} \cdot g_{g+}^{\downarrow} = -g_g^{-}$.*
4. *Π is rebuilt from it: Π itself supplies the second line (the vertical axis) and the direct exchange (multiplication by $g_g^{\downarrow}$, a rotation).*

Three objects, not pairwise equivalent



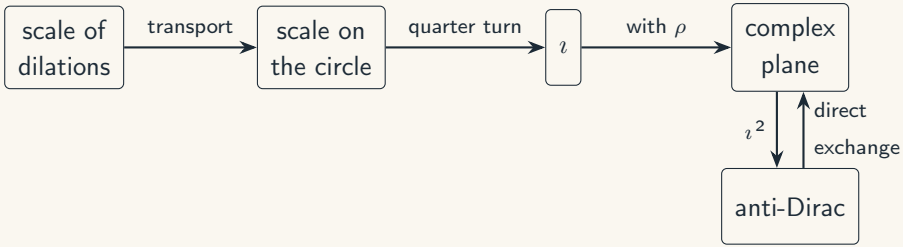
The two-level plane contains the other two. The anti-Dirac alone, carried by a line, is equivalent neither to Π nor to the complex plane.

Adjacency. Levels k steps apart would give $\omega_k = 2^{kn\infty}\, i$ and the same structure; adjacency fixes the scale factor.

The steps of the construction

Step	Given	Obtained
Dilations	grains of ratio 2^{n_∞}	exponential scale, positive values
Circle	2^{n_∞} equally spaced points	$P_a \cdot P_b = P_{a+b}$, periodic scale
Quarter turn	2^{n_∞} divisible by 4	$(g_g^{\downarrow})^2 = -g_f^{-}$, hence $g_g^{\downarrow} = \imath$
Plane	dilations and rotations combined	$z = \rho\, e^{\theta \imath} = x + y\,\imath$
Anti-Dirac	$\imath$ applied twice to the axis	two superposed, opposite lines
Converse	anti-Dirac, doubling, direct exchange	$J^2 = -1$, complex plane
Two levels	fine and coarse levels, ratio 2^{n_∞}	$\simeq \mathbb{C}$ by ν; carries an anti-Dirac

Conclusion



- The scale of dilations yields only positive quantities.
- On the circle it becomes periodic; the quarter turn lies on the grid and squares to the half turn: this is $\imath$.
- $\imath^2$ folds the horizontal axis onto itself with the opposite orientation: an anti-Dirac, of which the complex plane is the square root.

Thank you