Real Numbers as Stratified Objects

A pragmatic construction

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Outline

This talk: a fourth construction of the reals

Goal. Present a *pragmatic stratal construction* of the real numbers within the grain program, complementing the existing approaches.

Four constructions of $\mathbb{R}$.

1. *Algebraic construction* ($\mathbb{R}$ as a field).
2. *Classical constructions* (Dedekind cuts, Cauchy sequences).
3. *Non-standard construction* (developed within the grain program).
4. *Pragmatic stratal construction* (this talk).

What's new.

- Real numbers as *stratified objects* whose properties depend on the resolution level.
- A principle of *relativity of the status of reals*.
- Central mechanism: *stratal promotion* of a grain.

Compatibility. The pragmatic construction does not contradict the others: it enriches them with a stratal dimension.

Four constructions of the reals

The grain program now admits four complementary constructions of $\mathbb{R}$:

Algebraic construction. $\mathbb{R}$ as a field, via classical algebraic structures. The most abstract and universal.

Classical constructions.

- Dedekind cuts of $\mathbb{Q}$.
- Cauchy sequences of rationals.

Topological completions.

Non-standard construction. Developed within the grain program. The grains form a *ring* $\widetilde{\mathbb{R}}$ (units $=$ powers of 2); the field $\mathbb{R} = \widetilde{\mathbb{R}}_{\text{fin}}/\mathfrak{m}$ is recovered via the standard part operator. This is the bridge to classical analysis.

Pragmatic stratal construction (this talk). Reals as *families of stratal representations* at different resolution levels. Properties (measure, extension, cardinality) are *relative* to the observation level.

Why a pragmatic construction?

Classical perspective. Reals are absolute objects, independent of any representation. The algebraic approach makes them elements of an abstract structure (field).

Pragmatic perspective. A real number is always observed at a given resolution level. Its properties (measure, extension, cardinality) are relative to that level.

Epistemological motivation.

- A point grain of zero measure at level $-n_\infty$ is not a "substanceless" object.
- At level $-2n_\infty$, it becomes a grain of non-zero extension and measure.
- The apparent "disappearance" is a representation effect, not an intrinsic property.

Epistemological status. The pragmatic stratal construction is a *new object*, with no direct analog in classical approaches, but *compatible* with them.

Compatibility with classical approaches

No invalidation. None of the classical constructions is invalidated. The pragmatic construction enriches them with a stratal dimension.

Two bridges.

- The *standard part operation* (cf. notes on the standard part) connects the stratal family of a real to its canonical Dedekind cut.
- *Classical Cauchy sequences* correspond to *stratal sequences* of a real: each representative at level k is a term of the sequence.

Conceptual stance.

- Effective: explicit production of objects by a geometric mechanism.
- Operational: reals are defined by what one can do with them at each resolution level.

Choice of the standard

Initial calibration. We start with the unit circle C_1 of *circumference 1* (and thus radius $1/(2\pi)$).

Why circumference 1?

- Direct correspondence with the stratal decimal system.
- At resolution $-kn_\infty$, the circle contains 2^{kn_∞} grains of elementary thickness $1/2^{kn_\infty}$.
- The position ℓ is encoded by digits after the decimal point, without parasitic multiplicative factor.
- At level $-kn_\infty$, exactly kn_∞ digits appear after the point.

Strategy. Calibrate on the unit circle, extend to the diameter, then generalize to any stratifiable structure.

Stratification at resolution $-n_\infty$

Proposition (Coarse mosaic)

At resolution $-n_\infty$, the unit circle C_1 factorizes as

$$C_1 = \left(1,\ 2^{n_\infty},\ \frac{1}{2^{n_\infty}}\right).$$

The mosaic $\mathcal{B}_{-n_\infty} \simeq \mathcal{G}_{-n_\infty}$ contains 2^{n_∞} grains $b_{k,-n_\infty}$ $(k = 1, \ldots, 2^{n_\infty})$, each of measure $1/2^{n_\infty}$.

Status at this level.

- Each grain $b_{k,-n_\infty}$ appears as a *point*.
- Its measure $1/2^{n_\infty}$ is a level-n_∞ infinitesimal.
- Seen from standard resolution, it is “zero”.

Reading. A grain is point-like *from the perspective of its own resolution level*. The infinitesimal nature is invisible at this level.

Stratification at resolution $-2n_\infty$

Proposition (Fine mosaic)

At resolution $-2n_\infty$, the same circle C_1 factorizes as

$$C_1 = \left(1,\ 2^{2n_\infty},\ \frac{1}{2^{2n_\infty}}\right).$$

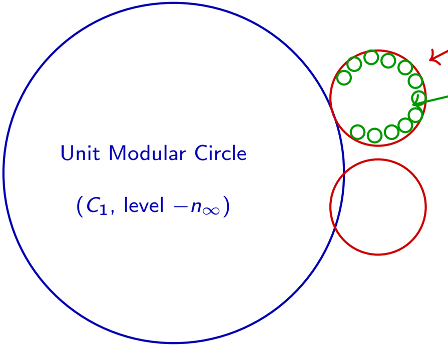
The fine mosaic $\mathcal{B}_{-2n_\infty}$ contains 2^{2n_∞} sub-grains of measure $1/2^{2n_\infty}$.
Each coarse grain $b_{k,-n_\infty}$ contains exactly 2^{n_∞} sub-grains of the fine mosaic:

$$b_{k,-n_\infty} = \{b^k_{1,-2n_\infty}, \dots, b^k_{\ell,-2n_\infty}, \dots, b^k_{2^{n_\infty},-2n_\infty}\}.$$

Status at this level.

- Each grain $b_{k,-n_\infty}$ becomes an *extended object*.
- Its measure is $\mu(b_{k,-n_\infty}) = 2^{n_\infty}/2^{2n_\infty} = 1/2^{n_\infty}$.
- This measure is *non-zero* at level $-2n_\infty$.

The graphical mechanism



$$b_{k,-n_\infty} = \{b^k_{1,-2n_\infty}, \ldots, b^k_{\ell,-2n_\infty} \ldots b^k_{2^{n_\infty},-2n_\infty}\}$$

$$\mu(b_{k,-n_\infty})/\mathcal{G}_{-2n_\infty}$$
$$= 0,0\ldots100\ldots0$$
($2n_\infty$ digits after the point)

Reading. The big circle is the unit circle at level $-n_\infty$. The two red circles "zoom" on a particular grain $b_{k,-n_\infty}$ seen at level $-2n_\infty$. Green circles depict the sub-grains.

The stratal decimal expansion

Proposition (Decimal expression of the stratal measure)

The measure of $b_{k,-n_\infty}$ in the resolution system $\mathcal{G}_{-2n_\infty}$ admits the binary expansion:

$$\frac{\mu(b_{k,-n_\infty})}{\mathcal{G}_{-2n_\infty}} = 0,0\ldots100\ldots0$$

with $2n_\infty$ digits after the point. The position of the digit "1" is determined by the index k and the cardinality of the fine mosaic.

Reading the expansion.

- First "0"s (before the "1"): position of the grain in the coarse mosaic (encoding at resolution $-n_\infty$).
- The "1": indication that the grain is present (non-zero).
- Final "0"s (after the "1"): internal structure of the grain at fine resolution (sub-grains).

Visibility by level.

- At level $-n_\infty$: only the first n_∞ digits visible (position, no extension).
- At level $-2n_\infty$: all $2n_\infty$ digits visible (position + internal structure).

The central theorem: stratal promotion

Theorem (Stratal promotion of a grain)

For every grain $b_{k,-n_\infty}$ of the coarse mosaic, passing from resolution $-n_\infty$ to resolution $-2n_\infty$ performs a *stratal promotion*:

$$\boxed{(\text{point grain, zero measure}) \xrightarrow{\text{passage } -n_\infty \to -2n_\infty} (\text{extended grain, non-zero measure}).}$$

Quantitative vs qualitative.

- Quantitative: measure $= 1/2^{n_\infty}$ *in both cases.*
- Qualitative: status changes from "standard zero" to "stratally non-zero".

Proof. Direct consequence of the two factorization propositions. $1/2^{n_\infty}$ is:

- A level-n_∞ infinitesimal at resolution $-n_\infty$ (matches the elementary thickness) — appears point-like.
- Non-zero at resolution $-2n_\infty$ (coarser than the elementary thickness $1/2^{2n_\infty}$) — appears extended.

Interpretation as change of representation

Crucial point. The stratal promotion is *not* an ontological transformation of the grain. The grain $b_{k,-n_\infty}$ does not "change" nature when passing from one resolution to another.

It is a change of representation.

- The same object is represented differently at each observation level.
- The intrinsic measure (computable at any sufficiently fine level) is $1/2^{n_\infty}$.
- Only its *appearance* (zero or non-zero) depends on the reference system.

General perspective. This reading is part of a broader principle: the *relativity of the status of reals*. The properties of a real number are not absolute; they depend on the resolution level at which one observes it.

Status of the construction.

- No phenomenological interpretation.
- Operational and effective.
- Compatible with classical objectivity (via standard part).

The general principle

Principle (Stratal relativity)

All properties of a real number — *measure*, *extension*, *cardinality*, and other metric or structural attributes — are *relative to the resolution level* at which one observes it.
No property is intrinsically absolute. A real number is not a fixed-property object, but a *family of stratal representations* at different levels.

Consequences.

- A real can have zero measure at one level and non-zero at another, without contradiction.
- A continuum can have different cardinalities according to the counting mode (positional or combinatorial).
- The extension of a grain depends on the observation level (point or extended).

Conceptual reading. Stratal relativity does not weaken the notion of real number — it *enriches* it by revealing its different facets at each level of analysis.

Compatibility with Dedekind

Proposition (Link with Dedekind)

The pragmatic stratal construction is compatible with Dedekind cuts. For any stratal family $\{b_{k,-mn_\infty}\}_{m\geq 1}$ associated with a real x, the *standard part* of the sequence of measures gives the canonical Dedekind cut:

$$\mathrm{st}\left(\bigcup_m b_{k,-mn_\infty}\right) = \{q \in \mathbb{Q} : q < x\}.$$

Reading. The classical cut is recovered as the *standard limit* of the stratal construction, with no contradiction with Dedekind's axioms.

Direction of generality.

- Stratal construction $\to$ classical (by standard part): *loses* the stratal structure.
- Classical $\to$ stratal: requires choosing a resolution level for each representative.

The stratal construction is *strictly finer*.

Compatibility with Cauchy

Proposition (Link with Cauchy)

The pragmatic stratal construction is compatible with Cauchy sequences. The stratal family $\{b_{k,-mn_\infty}\}_{m\geq 1}$ of a real x corresponds to a *stratal sequence of representatives*, where each grain at level $-mn_\infty$ is a term of the sequence.
The classical Cauchy condition is automatically satisfied: two representatives $b_{k,-mn_\infty}$ and $b_{k,-m'n_\infty}$ (for $m, m' \geq m_0$) are at infinitesimal distance.
The classical real x is recovered as the *standard part* of the stratal sequence.

Why automatic? The graded hierarchy of stratal infinitesimals guarantees that representatives at sufficiently fine levels are infinitesimally close — exactly the Cauchy condition.

Compatibility diagram

$$\text{Stratal construction} \longrightarrow \text{Standard part} \longrightarrow \text{Classical construction}$$

Stratal construction	Standard part	Classical construction
Family $\{b_{k,-mn_\infty}\}_m$	Operation st	Dedekind cut
Hierarchy preserved	Collapse to a single object	Cauchy sequence

Reading.

- The stratal construction is *strictly finer* than the classical: it preserves the structure of the stratal hierarchy.
- The standard part operation *reduces* this hierarchy to a unique object (the classical real).
- Information about resolution levels is irreversibly lost.

Consequence. Two different stratal families can have the same classical real as standard part. The classical reals are the "observable" projection of a richer structure.

Comparison with non-standard analysis

Compared to Robinson (1960).

Common point. Both work with infinitesimals and recover $\mathbb{R}$ as a *quotient by infinitesimals*: here $\mathbb{R} = \widetilde{\mathbb{R}}_{\mathrm{fin}}/\mathfrak{m}$ (the grains form a ring, the reals a field).

Difference 1 — Construction method.

- Robinson: ultraproducts and model theory.
- Stratal: *geometric hierarchization* of resolution levels.

Difference 2 — Level structure.

- Robinson: a *single* level of infinitesimals ($\mathbb{R}^*$).
- Stratal: a *graded hierarchy* ($\mathcal{G}_{-kn_\infty}$ for $k \geq 1$).

Difference 3 — Nature of the approach.

- Robinson: essentially *analytical*.
- Stratal: essentially *geometric and operational*.

Conclusion. The two approaches are *compatible and complementary*: this chapter is the bridge between the grain program and classical analysis.

From circle to diameter

Proposition (Extension to the diameter)

The pragmatic construction calibrated on the unit circle extends naturally to the *diameter of the circle* (segment of length $1/\pi$, or more generally a unit segment after normalization).
The stratal promotion mechanism applies *verbatim* to the diameter, replacing the cyclic structure with the linear structure.

Mosaic structure.

- Coarse mosaic: 2^{n_∞} grains of measure $1/2^{n_\infty}$.
- Fine mosaic: 2^{2n_∞} grains of measure $1/2^{2n_\infty}$.

Stratal promotion operates identically. Each grain of the coarse mosaic becomes an extended object containing 2^{n_∞} sub-grains at the fine mosaic level.

Structural lesson. Neither the cyclic structure of the circle nor the linear structure of the diameter is essential to the mechanism. What matters is the *factorization structure* of the stratification.

Generalization to any stratifiable structure

Proposition (Universal generalization)

The pragmatic stratal construction, calibrated on the unit circle and extended to the diameter, applies to *any stratifiable structure* of the grain program:

- The stratal plane (Volume II).
- Any stratal mosaic of the program.

The same stratal promotion mechanism operates: a point grain at a resolution level becomes an extended grain at the finer level — without ontological change, but by change of representation.

Universality.

- Not a particular mechanism for the circle or diameter.
- A general principle governing any stratifiable object.
- Circle and diameter are *initial calibration standards*.

Consistency with the program. A single structural mechanism (stratal multiplication) governs all the constructions of the grain program. The pragmatic construction is its expression in the domain of real numbers.

What is solid

Established results.

- A *rigorous calibration* on the unit circle of circumference 1.
- A *central mechanism* of stratal promotion (Theorem 1).
- A precise *algebraic status*: the grains form a *ring* $\widetilde{\mathbb{R}}$ (units = powers of 2), and $\mathbb{R} = \widetilde{\mathbb{R}}_{\mathrm{fin}}/\mathfrak{m}$ is a *field* — via density of grains (st surjective) $\Rightarrow$ maximality of $\mathfrak{m}$.
- An *explicit compatibility* with the classical constructions of Dedekind and Cauchy.
- A *universal generalization* to any stratifiable structure.

Synthesis. The pragmatic stratal construction completes, without contradicting, the algebraic, classical, and non-standard approaches.

The principal contribution is conceptual:

- Real numbers are conceived as *families of stratal representations*, not as fixed-property objects.
- All properties (measure, extension, cardinality) become *relative* to the observation level.
- This relativity does not weaken the notion of real number — it *enriches* it.

Open directions

Algebraic structure (now settled).

- The grains form a *ring* $\widetilde{\mathbb{R}}$ (commutative, integral); its only units are the powers of 2. *Not* a field.
- The field is the *promotion* $\mathbb{R} = \widetilde{\mathbb{R}}_{\mathrm{fin}}/\mathfrak{m}$ (finite part modulo the ideal $\mathfrak{m}$ of infinitesimals). The inverse of a non-unit (e.g. 3) exists only *after* this quotient.
- Proof route: density of grains (st surjective, e.g. by a Thales construction) $\Rightarrow$ $\mathfrak{m}$ maximal $\Rightarrow$ $\mathbb{R}$ a field. (The density lemma rests on the binary definition of grains.)

Order structure (open).

- Comparing two reals via their stratal representations; completeness of the induced order.

Topology (open).

- Natural topology on the space of stratal families; is it strictly finer than the classical topology of $\mathbb{R}$?

Links with the corpus.

- Topo-set-theoretic remarks; stratal rectification; stratal metric.

Concluding thoughts

A relative status, an enriched object.

The real numbers, classically conceived as absolute objects, acquire in the pragmatic stratal construction a *relative status*: their properties depend on the observation level.

This relativity is not subjective — it follows from the *geometric structure* of the grain program. Each resolution level $-kn_\infty$ defines a precise observation framework, with its elementary thickness, its cardinality, and its mode of reading the grains.

The central mechanism. Stratal promotion: a point grain becomes an extended grain at the finer level. Quantitatively conservative (measure preserved), qualitatively transformative (status changes). The same object, seen differently.

An enrichment, not a substitution. The pragmatic construction does not replace the algebraic, classical, or non-standard approaches. It adds a stratal dimension that reveals previously invisible structure of real numbers. The reals themselves form a *field* $\mathbb{R} = \widetilde{\mathbb{R}}_{\mathrm{fin}}/\mathfrak{m}$; the grains beneath them form a *ring*.

The unifying perspective. A single structural mechanism — stratification by multiplicative factorization — gives rise to all the constructions of the grain program, including this pragmatic

Thank you.

Questions?

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Grain Program

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